Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 5532691, 17 pages
https://doi.org/10.1155/2021/5532691

Research Article

The Galerkin Method for Fourth-Order Equation of the Moore–Gibson–Thompson Type with Integral Condition

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Received 25 February 2021; Revised 14 March 2021; Accepted 24 March 2021; Published 7 April 2021

Academic Editor: Kamyar Hosseini

In this manuscript, we consider the fourth order of the Moore–Gibson–Thompson equation by using Galerkin's method to prove the solvability of the given nonlocal problem.

1. Introduction

Research on the nonlinear propagation of sound in a situation of high amplitude waves has shown literature on physically well-founded partial differential models (see, e.g., [1–23]). This still very active field of research is carried by a wide range of applications such as the medical and industrial use of high-intensity ultrasound in lithotripsy, thermotherapy, ultrasound cleaning, and sonochemistry. The classical models of nonlinear acoustics are Kuznetsov's equation, the Westervelt equation, and the KZK (Kokhlov-Zabolotskaya-Kuznetsov) equation. For a mathematical existence and uniqueness analysis of several types of initial boundary value problems for these nonlinear second order in time PDEs, we refer to [24–44]. Focusing on the study of the propagation of acoustic waves, it should be noted that the MGT equation is one of the equations of nonlinear acoustics describing acoustic wave propagation in gases and liquids. The behavior of acoustic waves depends strongly on the medium property related to dispersion, dissipation, and nonlinear effects. It arises from modeling high-frequency ultrasound (HFU) waves (see [10, 12, 34]). The derivation of the equation, based on continuum and fluid mechanics, takes into account viscosity and heat conductivity as well as effect of the radiation of heat on the propagation of sound. The original derivation dates back to [44]. This model is realized through the third-order hyperbolic equation:

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = 0. \quad (1)$$

The unknown function $u = u(x, t)$ denotes the scalar acoustic velocity, c denotes the speed of sound, and τ denotes the thermal relaxation. Besides, the coefficient $b = \beta c^2$ is related to the diffusively of the sound with $\beta \in (0, \tau]$. In [44], Chen and Palmieri studied the blow-up result for the semilinear Moore–Gibson–Thompson equation with nonlinearity of derivative type in the conservative case defined as follows:

$$\beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u_t|^p, x \in \mathbb{R}^n, t > 0. \quad (2)$$

This paper is related to the following works (see [16, 39]). Now, when we talk about the (MGT) equation with memory term, we have Lasieka and Wang in [17] who studied the exponential decay of the energy of the temporally third-order (Moore–Gibson–Thompson) equation with a memory term as follows:

$$\tau u_{ttt} + \alpha u_{tt} - c^2 A u - b A u_t - \int_0^t g(t-s) A w(s) ds = 0, \quad (3)$$

where τ, α, b, c^2 are physical parameters and A is a positive self-adjoint operator on a Hilbert space H. The convolution term $\int_0^t g(t-s) A w(s) ds$ reflects the memory effects of materials due to viscoelasticity. In [18], Lasieka and Wang studied the general decay of solution of same problem above. The Moore–Gibson–Thompson equation with a nonlocal condition is a new posed problem. Existence and uniqueness of the generalized solution are established by using the Galerkin method. These problems can be encountered in many scientific domains and many engineering models (see previous works [5, 25–32, 35, 36, 40, 41]). Mesloub and Mesloub in [33] have applied the Galerkin method to a higher dimension mixed with nonlocal problem for a Boussinesq equation, while Boulaaras et al. investigated the Moore–Gibson–Thompson equation with the integral condition in [4]. Motivated by these outcomes, we improve the existence and uniqueness by the Galerkin method of the fourth-order equation of the Moore–Gibson–Thompson type with integral condition; this problem was cited by the work of Dell'Oro and Pata in [9].

We define the problem as follows:

$$\begin{cases} u_{tttt} + \alpha u_{ttt} + \beta u_{tt} - \rho \Delta u - \delta \Delta u_t - \gamma \Delta u_{tt} = 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), u_{tt}(x,0) = u_2(x), u_{ttt}(x,0) = u_3(x) \\ \dfrac{\partial u}{\partial \eta} = \displaystyle\int_0^t \int_\Omega u(\xi,\tau) d\xi d\tau, x \in \partial\Omega. \end{cases} \quad (4)$$

The aim of this manuscript is to consider the following nonlocal mixed boundary value problem for the Moore–Gibson–Thompson (MGT) equation for all $(x\,;t) \in Q_T = (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. solution of the posed problem.

We divide this paper into the following: In "Preliminaries," some definitions and appropriate spaces have been given. Then in "Solvability of the Problem," we use Galerkin's method to prove the existence, and in "Uniqueness of Solution," we demonstrate the uniqueness.

2. Preliminaries

Let $V(Q_T)$ and $W(Q_T)$ be the set spaces defined, respectively, by

$$\begin{gathered} V(Q_T) = \left\{ u \in W_2^1(Q_T) \colon u_t \in W_2^1(Q_T) \colon u_{tt} \in W_2^1(Q_T) \right\}, \\ W(Q_T) = \{ u \in V(Q_T) \colon u(x,T) = 0 \}. \end{gathered} \quad (5)$$

Consider the equation

$$\begin{aligned} &(u_{tttt}, v)_{L^2(Q_T)} + \alpha(u_{ttt}, v)_{L^2(Q_T)} + \beta(u_{tt}, v)_{L^2(Q_T)} - \varrho(\Delta u, v)_{L^2(Q_T)} \\ &\quad - \delta(\Delta u_t, v)_{L^2(Q_T)} \gamma(\Delta u_{tt}, v)_{L^2(Q_T)} = 0, \end{aligned} \quad (6)$$

where $(.,.)_{L^2(Q_T)}$ depend on the inner product in $L^2(Q_T)$, u is supposed to be a solution of (1), and $v \in W(Q_T)$. Upon using (6) and (1), we find

$$\begin{aligned} &-(u_{ttt}, v_t)_{L^2(Q_T)} - \alpha(u_{tt}, v_t)_{L^2(Q_T)} - \beta(u_t, v_t)_{L^2(Q_T)} + \varrho(\nabla u, \nabla v)_{L^2(Q_T)} \\ &\quad + \delta(\nabla u_t, \nabla v)_{L^2(Q_T)} - \gamma(\nabla u_t, \nabla v_t)_{L^2(Q_T)} \\ &\quad = \varrho \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_\Omega u(\xi,\tau) d\xi d\tau \right) ds_x dt + \delta \int_0^T \int_{\partial\Omega} v \int_\Omega u(\xi,t) d\xi ds_x dt \\ &\quad - \delta \int_0^T \int_{\partial\Omega} v \int_\Omega u_0(\xi) d\xi ds_x dt - \gamma \int_0^T \int_{\partial\Omega} v_t \left(\int_0^t \int_\Omega u_\tau(\xi,\tau) d\xi d\tau \right) ds_x dt \\ &\quad + (u_3(x), v(x,0))_{L^2(\Omega)} + \alpha(u_2(x), v(x,0))_{L^2(\Omega)} + \beta(u_1(x), v(x,0))_{L^2(\Omega)} \\ &\quad - \gamma(\Delta u_1, v(x,0))_{L^2(\Omega)}. \end{aligned} \quad (7)$$

Now, we give two useful inequalities:

(i) Gronwall inequality: if for any $t \in I$, we have

$$y(t) \le h(t) + c \int_0^t y(s) ds, \quad (8)$$

where $h(t)$ and $y(t)$ are two nonnegative integrable functions on the interval I with $h(t)$ nondecreasing and c is constant, then

$$y(t) \le h(t) \exp(ct) \quad (9)$$

(ii) Trace inequality: when $w \in W_1^2(\Omega)$, we have

$$\|w\|_{L^2(\partial\Omega)}^2 \le \varepsilon \|\nabla w\|_{L^2(\Omega)}^2 + l(\varepsilon)\|w\|_{L^2(\Omega)}^2, \quad (10)$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $l(\varepsilon)$ is a positive constant.

Definition 1. If a function $u \in V(Q_T)$ satisfies Equation (3), each $v \in W(Q_T)$ is called a generalized solution of problem (1).

3. Solvability of the Problem

Here, by using Galerkin's method, we give the existence of problem (1).

Theorem 2. *If $u_0 \in$, $u_1 \in$ and $u_2 \in$, $u_3 \in$, then there is at least one generalized solution in $V(Q_T)$ to problem (1).*

Proof. Let $\{Z_k(x)\}_{k \ge 1}$ be a fundamental system in $W_2^1(\Omega)$, such that $(Z_k, Z_l)_{L^2(\Omega)} = \delta_{k,l}$. Now, we will find an approximate solution of the problem (1) in the form

$$u^N(x,t) = \sum_{k=1}^N C_k(t) Z_k(x), \quad (11)$$

where the constants $C_k(t)$ are defined by the conditions

$$C_k(t)=\left(u^N(x,t),Z_k(x)\right)_{L^2(\Omega)},\ k=1,\cdots,N, \quad (12)$$

and can be determined from the relations

$$\begin{aligned}&\left(u^N_{tttt},Z_l(x)\right)_{L^2(\Omega)}+\alpha\left(u^N_{ttt},Z_l(x)\right)_{L^2(\Omega)}+\beta\left(u^N_{tt},Z_l(x)\right)_{L^2(\Omega)}\\&\quad+\varrho\left(\nabla u^N,\nabla Z_l(x)\right)_{L^2(\Omega)}+\delta\left(\nabla u^N_t,\nabla Z_l(x)\right)_{L^2(\Omega)}+\gamma\left(\nabla u^N_{tt},\nabla Z_l(x)\right)_{L^2(\Omega)}\\&=\varrho\int_{\partial\Omega}Z_l(x)\left(\int_0^t\int_\Omega u^N(\xi,\tau)d\xi d\tau\right)ds_x+\delta\int_{\partial\Omega}Z_l(x)\left(\int_0^t\int_\Omega u^N_\tau(\xi,\tau)d\xi d\tau\right)ds_x\\&\quad+\gamma\int_{\partial\Omega}Z_l(x)\left(\int_0^t\int_\Omega u^N_{\tau\tau}(\xi,\tau)d\xi d\tau\right)ds_x.\end{aligned} \quad (13)$$

Invoking to (11) in (6) gives for $l=1,\cdots,N$.

$$\begin{aligned}&\int_\Omega\sum_{k=1}^N\Big\{C_k'''(t)Z_k(x)Z_l(x)+\alpha C_k''(t)Z_k(x)Z_l(x)+\beta C_k''(t)Z_k(x)Z_l(x)\\&\quad+\varrho C_k(t)\nabla Z_k(x).\nabla Z_l(x)+\delta C_k'(t)\nabla Z_k(x).\nabla Z_l(x)+\gamma C_k'(t)\nabla Z_k.\nabla Z_l\Big\}dx\\&=\varrho\sum_{k=1}^N\int_0^t C_k(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau+\delta\sum_{k=1}^N\int_0^t C_k'(\tau)\\&\quad\times\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau+\gamma\sum_{k=1}^N\int_0^t\left(C_k'(\tau)\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau.\end{aligned} \quad (14)$$

From (7), it follows that

$$\begin{aligned}&\sum_{k=1}^N C_k'''(t)\left(Z_k(x),Z_l(x)\right)_{L^2(\Omega)}+\alpha C_k''(t)\left(Z_k(x),Z_l(x)\right)_{L^2(\Omega)}\\&\quad+\beta C_k'(t)\left(Z_k(x),Z_l(x)\right)_{L^2(\Omega)}+\varrho C_k(t)\left(\nabla Z_k,\nabla Z_l\right)_{L^2(\Omega)}\\&\quad+\delta C_k'(t)\left(\nabla Z_k(x),\nabla Z_l(x)\right)_{L^2(\Omega)}+\gamma C_k''(t)\left(\nabla Z_k(x),\nabla Z_l(x)\right)_{L^2(\Omega)}\\&=\varrho\sum_{k=1}^N\int_0^t C_k(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau\\&\quad+\delta\sum_{k=1}^N\int_0^t C_k'(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau\\&\quad+\gamma\sum_{k=1}^N\int_0^t\left(C_k'(\tau)\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds\right)d\tau,\ l=1,\cdots,N.\end{aligned} \quad (15)$$

Let

$$\begin{aligned}&\left(Z_k,Z_l\right)_{L^2(\Omega)}=\delta_{kl}=\begin{cases}1,\ k=l\\0,\ k\neq l,\end{cases}\\&\left(\nabla Z_k,\nabla Z_l\right)_{L^2(\Omega)}=\gamma_{kl},\\&\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds=\chi_{kl}.\end{aligned} \quad (16)$$

Then (8) can be written as

$$\begin{aligned}&\sum_{k=1}^N C_k'''(t)\delta_{kl}+\alpha C_k''(t)\delta_{kl}+C_k'(t)\left(\beta\delta_{kl}+\gamma\gamma_{kl}\right)+\delta C_k'(t)\gamma_{kl}\\&\quad+\varrho C_k(t)\gamma_{kl}-\int_0^t\Big(\varrho C_k(\tau)\chi_{kl}+\delta C_k'(\tau)\chi_{kl}+\gamma C_k''(\tau)\chi_{kl}=0.\end{aligned} \quad (17)$$

A differentiation with respect to t yields

$$\begin{aligned}&\sum_{k=1}^N C_k''''(t)\delta_{kl}+\alpha C_k'''(t)\delta_{kl}+C_k''(t)\left(\beta\delta_{kl}+\gamma\gamma_{kl}\right)+C_k''(t)\left(\delta\gamma_{kl}-\gamma\chi_{kl}\right)+C_k'(t)\left(\varrho\gamma_{kl}-\delta\chi_{kl}\right)\gamma_{kl}-\varrho C_k(t)\chi_{kl}=0,\\&\begin{cases}\sum_{k=1}^N\left[C_k'''(0)\delta_{kl}+\alpha C_k''(0)\delta_{kl}+C_k'(0)\left(\beta\delta_{kl}+\gamma\gamma_{kl}\right)+\delta C_k'(0)\gamma_{kl}+\varrho C_k(0)\gamma_{kl}\right]=0\\C_k(0)=\left(Z_k,u_0\right)_{L^2(\Omega)},\ C_k'(0)=\left(Z_k,u_1(x)\right)_{L^2(\Omega)},\ C_k''(t)=\left(Z_k,u_2(x)\right)_{L^2(\Omega)},C_k'''(t)=\left(Z_k,u_3(x)\right)_{L^2(\Omega)}.\end{cases}\end{aligned} \quad (18)$$

Thus, for every n, there exists a function $u^N(x)$ satisfying (6). Now, we will demonstrate that the sequence u^N is bounded. To do this, we multiply each equation of (6) by the appropriate $C_k'(t)$ summing over k from 1 to N then integrating the resultant equality with respect to t from 0 to τ, with $\tau\leq T$, which yields

$$\begin{aligned}&\left(u^N_{tttt},u^N_t\right)_{L^2(Q_\tau)}+\alpha\left(u^N_{ttt},u^N_t\right)_{L^2(Q_\tau)}+\beta\left(u^N_{tt},u^N_t\right)_{L^2(Q_\tau)}+\varrho\left(\nabla u^N,\nabla u^N_t\right)_{L^2(Q_\tau)}\\&\quad+\delta\left(\nabla u^N_t,\nabla u^N_t\right)_{L^2(Q_\tau)}+\gamma\left(\nabla u^N_{tt},\nabla u^N_t\right)_{L^2(Q_\tau)}=\varrho\int_0^\tau\int_{\partial\Omega}u^N_t(x,t)\\&\quad\times\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt+\delta\int_0^\tau\int_{\partial\Omega}u^N_t(x,t)\left(\int_0^t\int_\Omega u^N_t(\xi,\eta)d\xi d\eta\right)ds_x dt\\&\quad+\gamma\int_0^\tau\int_{\partial\Omega}u^N_t(x,t)\left(\int_0^t\int_\Omega u^N_{tt}(\xi,\eta)d\xi d\eta\right)ds_x dt.\end{aligned} \quad (19)$$

After simplification of the LHS of (19), we observe that

$$\begin{aligned}\left(u^N_{tttt},u^N_t\right)_{L^2(Q_\tau)}&=-\int_0^\tau\left(u^N_{ttt},u^N_{tt}\right)_{L^2(\Omega)}dt\\&\quad+\left(u^N_{\tau\tau\tau}(x,\tau),u^N_\tau(x,\tau)\right)_{L^2(\Omega)}\\&\quad-\left(u^N_{ttt}(x,0),u^N_t(x,0)\right)_{L^2(\Omega)},\end{aligned} \quad (20)$$

$$\begin{cases}\alpha\left(u^N_{ttt},u^N_t\right)_{L^2(Q_\tau)}=\alpha\left(u^N_{\tau\tau}(x,\tau),u^N_\tau(x,\tau)\right)_{L^2(\Omega)}-\left(u^N_{tt}(x,0),u^N_t(x,0)\right)_{L^2(\Omega)}\\-\alpha\int_0^\tau\|u_{tt}(x,t)\|^2_{L^2(\Omega)}dt,\end{cases} \quad (21)$$

$$\beta\left(u_{tt}^N, u_t^N\right)_{L^2(Q_\tau)} = \frac{\beta}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\beta}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{22}$$

$$\varrho\left(\nabla u^N, \nabla u_t^N\right)_{L^2(Q_\tau)} = \frac{\varrho}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\varrho}{2}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{23}$$

$$\delta\left(\nabla u_t^N, \nabla u_t^N\right)_{L^2(Q_\tau)} = \delta\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{24}$$

$$\gamma\left(\nabla u_{tt}^N, \nabla u_t^N\right)_{L^2(Q_\tau)} = \frac{\gamma}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\gamma}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{25}$$

$$\begin{aligned}\varrho\int_0^\tau\int_{\partial\Omega} u_t^N\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt &= \varrho\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad - \varrho\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x,\end{aligned} \tag{26}$$

$$\begin{aligned}&\delta\int_0^\tau\int_{\partial\Omega} u_t^N\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad = \delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad\quad - \delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x,\end{aligned} \tag{27}$$

$$\begin{aligned}&\gamma\int_0^\tau\int_{\partial\Omega} u_t^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad = \gamma\int_0^\tau\int_{\partial\Omega} u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,t)d\xi\right)ds_x dt \\ &\quad\quad - \gamma\int_0^\tau\int_{\partial\Omega} u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,0)d\xi\right)ds_x dt.\end{aligned} \tag{28}$$

Taking into account the equalities (20) and (21) in (12), we obtain

$$\begin{aligned}&\left(u_{\tau\tau\tau}^N(x,\tau), u_\tau^N(x,\tau)\right)_{L^2(\Omega)} + \alpha\left(u_{\tau\tau}^N(x,\tau), u_\tau^N(x,\tau)\right)_{L^2(\Omega)} \\ &\quad + \frac{\beta}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{\varrho}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{\gamma}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &= \left(u_{ttt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)} + \alpha\left(u_{tt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)} \\ &\quad + \frac{\beta}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\varrho}{2}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\gamma}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^\tau\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(\Omega)}dt + \alpha\int_0^\tau\left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}^2 dt - \delta\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\ &\quad + \varrho\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x - \varrho\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad + \delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x - \delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x \\ &\quad + \gamma\int_0^\tau\int_{\partial\Omega} u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,t)d\xi\right)ds_x dt - \gamma\int_0^\tau\int_{\partial\Omega} u_t^N(x,t) \\ &\quad \times\left(\int_\Omega u_t^N(\xi,0)d\xi\right)ds_x dt.\end{aligned} \tag{29}$$

Now, multiplying each equation of (6) by the appropriate $C_k''(t)$, we add them up from 1 to N and then integrate with respect to t from 0 to τ, with $\tau \le T$, and we obtain

$$\begin{aligned}&\left(u_{tttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} + \alpha\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} + \beta\left(u_{tt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} \\ &\quad + \varrho\left(\nabla u^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} + \delta\left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} + \gamma\left(\nabla u_{tt}^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} \\ &\quad = \varrho\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad\quad + \delta\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad\quad + \gamma\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt.\end{aligned} \tag{30}$$

With the same reasoning in (12), we find

$$\begin{aligned}\left(u_{tttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} &= -\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \left(u_{\tau\tau\tau}^N(x,\tau), u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)} \\ &\quad - \left(u_{ttt}^N(x,0), u_{tt}^N(x,0)\right)_{L^2(\Omega)},\end{aligned} \tag{31}$$

$$\alpha\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} = \frac{\alpha}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\alpha}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{32}$$

$$\beta\left(u_{tt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} = \beta\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{33}$$

$$\begin{aligned}\varrho\left(\nabla u^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= \varrho\left(\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau)\right)_{L^2(Q_\tau)} \\ &\quad - \varrho\left(\nabla u^N(x,0), \nabla u_t^N(x,0)\right)_{L^2(\Omega)} \\ &\quad - \varrho\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt,\end{aligned} \tag{34}$$

$$\delta\left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} = \frac{\delta}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\delta}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{35}$$

$$\gamma\left(\nabla u_{tt}^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} = \gamma\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{36}$$

$$\begin{aligned}&\varrho\int_0^\tau\int_{\partial\Omega} u_{tt}^N\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad = \varrho\int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad\quad - \varrho\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x,\end{aligned} \tag{37}$$

$$\begin{aligned}&\delta\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad = \delta\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x - \delta\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x \\ &\quad\quad - \delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds,\end{aligned} \tag{38}$$

$$\gamma\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt$$
$$= \gamma\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x - \gamma\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x$$
$$- \gamma\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dt ds, \quad (39)$$

Upon using (31) and (32) into (23), we have

$$\left(u_{\tau\tau\tau}^N(x,\tau), u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)} + \frac{\alpha}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{\delta}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$+ \varrho\left(\nabla u^N(x,\tau),\nabla u_\tau^N(x,\tau)\right)_{L^2(\Omega)} = \int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+ \left(u_{ttt}^N(x,0), u_{tt}^N(x,0)\right)_{L^2(\Omega)} + \frac{\alpha}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2 - \beta\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+ \varrho\left(\nabla u^N(x,0),\nabla u_t^N(x,0)\right)_{L^2(\Omega)} + \varrho\int_0^\tau\left\|\nabla u_t(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+ \frac{\delta}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2 - \gamma\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+ \varrho\int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$- \varrho\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x + \delta\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x$$
$$- \delta\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x - \delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds_x$$
$$+ \gamma\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x - \gamma\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x$$
$$- \gamma\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dt ds_x. \quad (40)$$

Now, multiplying each equation of (6) by the appropriate $C_k'(t)$, we add them up from 1 to N and then integrate with respect to t from 0 to τ, with $\tau \le T$, and we obtain

$$\left(u_{tttt}^N, u_{ttt}^N\right)_{L^2(Q_\tau)} + \alpha\left(u_{ttt}^N, u_{ttt}^N\right)_{L^2(Q_\tau)} + \beta\left(u_{tt}^N, u_{ttt}^N\right)_{L^2(Q_\tau)} + \varrho\left(\nabla u^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)}$$
$$+ \delta\left(\nabla u_t^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)} + \gamma\left(\nabla u_{tt}^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)}$$
$$= \varrho\int_0^\tau\int_{\partial\Omega} u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt$$
$$+ \delta\int_0^\tau\int_{\partial\Omega} u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt$$
$$+ \gamma\int_0^\tau\int_{\partial\Omega} u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt. \quad (41)$$

With the same reasoning in (12), we find

$$\left(u_{tttt}^N, u_{ttt}^N\right)_{L^2(Q_\tau)} = \frac{1}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{1}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2, \quad (42)$$

$$\alpha\left(u_{ttt}^N, u_{ttt}^N\right)_{L^2(Q_\tau)} = \alpha\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2, \quad (43)$$

$$\beta\left(u_{tt}^N, u_{ttt}^N\right)_{L^2(Q_\tau)} = \frac{\beta}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\beta}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \quad (44)$$

$$\varrho\left(\nabla u^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)} = \varrho\left(\nabla u^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}$$
$$- \varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}$$
$$- \varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)} dt, \quad (45)$$

$$\delta\left(\nabla u_t^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)} = -\delta\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \delta\left(\nabla u_\tau^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}$$
$$- \delta\left(\nabla u_t^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}, \quad (46)$$

$$\gamma\left(\nabla u_{tt}^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)} = \frac{\gamma}{2}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\gamma}{2}\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \quad (47)$$

$$\varrho\int_0^\tau\int_{\partial\Omega} u_{ttt}^N\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt$$
$$= \varrho\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$- \varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x, \quad (48)$$

$$\delta\int_0^\tau\int_{\partial\Omega} u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt$$
$$= \delta\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x - \delta\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$
$$- \delta\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds, \quad (49)$$

$$\gamma\int_0^\tau\int_{\partial\Omega} u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt$$
$$= \gamma\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x - \gamma\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x$$
$$- \gamma\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dt ds, \quad (50)$$

A substitution of equalities (42) and (43) in (34) gives

$$\frac{1}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{\beta}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \varrho\left(\nabla u^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}$$
$$+\delta\left(\nabla u_\tau^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)} + \frac{\gamma}{2}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$= \frac{1}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2 - \alpha\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2 - \frac{\beta}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)} + \varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)} dt$$
$$+\delta\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \delta\left(\nabla u_t^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)} - \frac{\gamma}{2}\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\varrho\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x - \varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$+\delta\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x - \delta\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$

$$-\delta\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dtds\gamma+\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x$$
$$-\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x-\gamma\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dtds. \tag{51}$$

Multiplying (22) by λ_1, (33) by λ_2, and (44) by λ_3, we get

$$\lambda_1\left(u_{\tau\tau\tau}^N(x,\tau),u_\tau^N(x,\tau)\right)_{L^2(\Omega)}+\lambda_1\alpha\left(u_{\tau\tau}^N(x,\tau),u_\tau^N(x,\tau)\right)_{L^2(\Omega)}+\frac{\lambda_1\beta}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$+\frac{\lambda_1\varrho}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2+\left(\frac{\lambda_1\gamma}{2}+\frac{\lambda_2\delta}{2}\right)\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2+\lambda_2\left(u_{\tau\tau\tau}^N(x,\tau),u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}$$
$$+\left(\frac{\lambda_2\alpha}{2}+\frac{\lambda_3\beta}{2}\right)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+\lambda_2\varrho\left(\nabla u^N(x,\tau),\nabla u_\tau^N(x,\tau)\right)_{L^2(\Omega)}+\frac{\lambda_3}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$+\lambda_3\varrho\left(\nabla u^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}+\lambda_3\delta\left(\nabla u_\tau^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}+\frac{\lambda_3\gamma}{2}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$=\lambda_1\left(u_{ttt}^N(x,0),u_t^N(x,0)\right)_{L^2(\Omega)}+\lambda_1\alpha\left(u_{tt}^N(x,0),u_t^N(x,0)\right)_{L^2(\Omega)}+\frac{\lambda_1\beta}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\frac{\lambda_1\varrho}{2}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2+\left(\frac{\lambda_1\gamma}{2}+\frac{\lambda_2\delta}{2}\right)\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2+\lambda_1\int_0^\tau\left(u_{ttt}^N,u_{tt}^N\right)_{L^2(\Omega)}dt$$
$$+(\lambda_1\alpha-\lambda_2\beta)\int_0^\tau\|u_{tt}(x,t)\|_{L^2(\Omega)}^2dt+(\lambda_2\varrho-\lambda_1\delta)\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt$$
$$+(\lambda_2-\lambda_3\alpha)\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2dt+\lambda_2\left(u_{ttt}^N(x,0),u_{tt}^N(x,0)\right)_{L^2(\Omega)}+\left(\frac{\lambda_2\alpha}{2}-\frac{\lambda_3\beta}{2}\right)\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\lambda_2\varrho\left(\nabla u^N(x,0),\nabla u_t^N(x,0)\right)_{L^2(\Omega)}+(\lambda_3\delta-\lambda_2\gamma)\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt+\frac{\lambda_3}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\lambda_3\varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}+\lambda_3\varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)}dt,$$
$$+\lambda_3\delta\left(\nabla u_t^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}-\frac{\lambda_3\gamma}{2}\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\lambda_1\varrho\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dtds_x-\lambda_1\varrho\int_{\partial\Omega}\int_0^\tau u^N(x,t)\cdot\int_\Omega u^N(\xi,t)d\xi dtds_x$$
$$+(\lambda_1\delta-\lambda_2\varrho)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dtds_x-\lambda_1\delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\cdot\int_\Omega u^N(\xi,0)d\xi dtds_x$$
$$+(\lambda_1\gamma-\lambda_2\delta)\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,t)d\xi\right)ds_xdt-\lambda_1\gamma\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\cdot\left(\int_\Omega u_t^N(\xi,0)d\xi\right)ds_xdt.$$
$$+\lambda_2\varrho\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dtds_x+\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x$$
$$-\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x+\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x$$
$$-\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x-\lambda_2\gamma\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dtds_x$$
$$+\lambda_3\varrho\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dtds_x-\lambda_3\varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)d\xi dtds_x$$
$$+\lambda_3\delta\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x-\lambda_3\delta\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$
$$-\lambda_3\delta\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dtds+\lambda_3\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x$$
$$-\lambda_3\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x-\lambda_3\gamma\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dtds. \tag{52}$$

We can estimate all the terms in the right-hand side of (45) as follows:

$$\lambda_1\varrho\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dtds_x$$
$$\le\frac{\lambda_1\varrho}{2\varepsilon_1}\left(\varepsilon\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)+\frac{\lambda_1\varrho}{2}\varepsilon_1T|\Omega||\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,$$
$$-\lambda_1\varrho\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)d\xi dtds_x$$
$$\le\frac{\lambda_1\varrho}{2}\varepsilon\int_0^\tau\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2dt+\frac{\lambda_1\varrho}{2}\left(l(\varepsilon)+|\Omega||\partial\Omega|\right)\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,$$
$$(\lambda_1\delta-\lambda_2\varrho)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dtds_x$$
$$\le\frac{(\lambda_1\delta+\lambda_2\varrho)}{2\left(\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt+l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt\right)}$$

$$+\frac{(\lambda_1\delta+\lambda_2\varrho)}{2}|\Omega||\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt,$$

$$-\lambda_1\delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x$$
$$\le\frac{\lambda_1\delta}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt+l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt\right)$$
$$+\frac{\lambda_1\delta}{2}|\Omega||\partial\Omega|T\left\|u^N(x,0)\right\|^2_{L^2(\Omega)},$$

$$\lambda_2\varrho\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{\lambda_2\varrho}{2}\left(\frac{\varepsilon}{\varepsilon_2}\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}+\frac{l(\varepsilon)}{\varepsilon_2}\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_2\varrho}{2}\varepsilon_2|\Omega||\partial\Omega|T\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt,$$

$$\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x$$
$$\le\frac{\lambda_2\delta}{2\varepsilon_3}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_2\delta}{2}\varepsilon_3|\Omega||\partial\Omega|\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)},$$

$$-\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$
$$\le\frac{\lambda_2\delta}{2\varepsilon_4}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_2\delta}{2}\varepsilon_4|\Omega||\partial\Omega|\left\|u^N(x,0)\right\|^2_{L^2(\Omega)},$$

$$(\lambda_1\gamma-\lambda_2\delta)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{(\lambda_1\gamma+\lambda_2\delta)}{2}\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt$$
$$+\frac{(\lambda_1\gamma+\lambda_2\delta)}{2}\left(l(\varepsilon)+|\Omega||\partial\Omega|\right)\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt, \quad (53)$$

$$-\lambda_1\gamma\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,0)d\xi\right)ds_x dt$$
$$\le\frac{\lambda_1\gamma}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt+l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt\right)$$
$$+\frac{\lambda_1\gamma}{2}|\Omega||\partial\Omega|T\left\|u_t^N(x,0)\right\|^2_{L^2(\Omega)},$$

$$\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x$$
$$\le\frac{\lambda_2\gamma}{2\varepsilon_5}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_2\gamma}{2}\varepsilon_5|\Omega||\partial\Omega|\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)},-\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x$$
$$\le\frac{\lambda_2\gamma}{2\varepsilon_6}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_2\gamma}{2}\varepsilon_6|\Omega||\partial\Omega|\left\|u_t^N(x,0)\right\|^2_{L^2(\Omega)},$$

$$-\lambda_2\gamma\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{\lambda_2\gamma}{2}\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt+\frac{\lambda_2\gamma}{2}l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt$$
$$+\frac{\lambda_2\gamma}{2}|\Omega||\partial\Omega|\int_0^\tau\left\|u_{tt}^N(x,t)\right\|^2_{L^2(\Omega)}dt,$$

$$\lambda_3\varrho\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{\lambda_3\varrho}{2}\left(\frac{\varepsilon}{\varepsilon_7}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|^2_{L^2(\Omega)}+\frac{l(\varepsilon)}{\varepsilon_7}\left\|u_{\tau\tau}^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_3\varrho}{2}\varepsilon_7|\Omega||\partial\Omega|T\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt,$$

$$-\lambda_3\varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{\lambda_3\varrho}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|^2_{L^2(\Omega)}dt+l(\varepsilon)\int_0^\tau\left\|u_{tt}^N(x,t)\right\|^2_{L^2(\Omega)}dt\right)$$
$$+\frac{\lambda_3\varrho}{2}|\Omega||\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt,$$

$$\lambda_3\delta\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x$$
$$\le\frac{\lambda_3\delta}{2\varepsilon_8}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_3\delta}{2}\varepsilon_8|\Omega||\partial\Omega|\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)}, \quad (54)$$

$$-\lambda_3\delta\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$
$$\le\frac{\lambda_3\delta}{2\varepsilon_9}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{\lambda_3\delta}{2}\varepsilon_9|\Omega||\partial\Omega|\left\|u^N(x,0)\right\|^2_{L^2(\Omega)}, \quad (55)$$

$$-\lambda_3\delta\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{\lambda_3\delta}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|^2_{L^2(\Omega)}dt+l(\varepsilon)\int_0^\tau\left\|u_{tt}^N(x,t)\right\|^2_{L^2(\Omega)}dt\right)$$
$$+\frac{\lambda_3\delta}{2}|\Omega||\partial\Omega|\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt,$$

$$\lambda_3\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)d\xi ds_x \le \frac{\lambda_3\gamma}{2\varepsilon_{10}}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) + \frac{\lambda_3\gamma}{2}\varepsilon_{10}|\Omega||\partial\Omega|\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2, \tag{56}$$

$$-\lambda_3\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x \le \frac{\lambda_3\gamma}{2\varepsilon_{11}}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) + \frac{\lambda_3\gamma}{2}\varepsilon_{11}|\Omega||\partial\Omega|\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$-\lambda_3\gamma\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_{tt}^N(\xi,t)d\xi dt ds_x \le \frac{\lambda_3\gamma}{2}\varepsilon\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt + \frac{\lambda_3\gamma}{2}(l(\varepsilon)+|\Omega||\partial\Omega|)\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt,$$

$$-\frac{\lambda_1}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_1}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le \lambda_1\left(u_{\tau\tau\tau}^N(x,\tau),u_\tau^N(x,\tau)\right)_{L^2(\Omega)}, \tag{57}$$

$$-\frac{\lambda_2}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le \lambda_2\left(u_{\tau\tau\tau}^N(x,\tau),u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)},$$

$$-\frac{\lambda_1\alpha}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_1\alpha}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le \lambda_1\alpha\left(u_{\tau\tau}^N(x,\tau),u_\tau^N(x,\tau)\right)_{L^2(\Omega)},$$

$$-\frac{\lambda_2\varrho\varepsilon_{12}}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2\varrho}{2\varepsilon_{12}}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le \lambda_2\varrho\left(\nabla u^N(x,\tau),\nabla u_\tau^N(x,\tau)\right)_{L^2(\Omega)},$$

$$-\frac{\lambda_2\varrho\varepsilon_{13}}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2\varrho}{2\varepsilon_{13}}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le \lambda_3\varrho\left(\nabla u^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)},$$

$$-\frac{\lambda_3\delta\varepsilon_{14}}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_3\delta\varepsilon_{14}}{2}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le \lambda_3\delta\left(\nabla u_\tau^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)},$$

$$\lambda_1\left(u_{ttt}^N(x,0),u_t^N(x,0)\right)_{L^2(\Omega)} \le \frac{\lambda_1}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_1}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$\lambda_1\alpha\left(u_{tt}^N(x,0),u_t^N(x,0)\right)_{L^2(\Omega)} \le \frac{\lambda_1\alpha}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_1\alpha}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$\lambda_2\left(u_{ttt}^N(x,0),u_{tt}^N(x,0)\right)_{L^2(\Omega)} \le \frac{\lambda_2}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_2}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$\lambda_2\varrho\left(\nabla u^N(x,0),\nabla u_t^N(x,0)\right)_{L^2(\Omega)} \le \frac{\lambda_2}{2}\varrho\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_2}{2}\varrho\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$\lambda_3\varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)} \le \frac{\lambda_3}{2}\varrho\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_3}{2}\varrho\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$\lambda_3\delta\left(\nabla u_t^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)} \le \frac{\lambda_3}{2}\delta\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_3}{2}\delta\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{58}$$

$$\lambda_1\int_0^\tau\left(u_{ttt}^N,u_{tt}^N\right)_{L^2(\Omega)}dt \le \frac{\lambda_1}{2}\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2dt + \frac{\lambda_1}{2}\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt, \tag{59}$$

$$\lambda_3\varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)}dt \le \frac{\lambda_3\varrho}{2}\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt + \frac{\lambda_3\varrho}{2}\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt. \tag{60}$$

Combining inequalities (46)–(79) and equality (45) and making use of the following inequality:

$$m_1\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le m_1\left\|u^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_1\left\|u_t^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_1\left\|u^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$m_2\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le m_2\left\|u_t^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_2\left\|u_{tt}^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_2\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$m_3\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le m_3\left\|u_{tt}^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_3\left\|u_{ttt}^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_3\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$m_4\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le m_4\left\|\nabla u^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_4\left\|\nabla u_t^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_4\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2,$$

$$m_5\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \le m_5\left\|\nabla u_t^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_5\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(Q_\tau)}^2 + m_5\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{61}$$

where

$$m_1 = \frac{\lambda_2\delta}{2}\varepsilon_3|\Omega||\partial\Omega| + \frac{\lambda_3\delta}{2}\varepsilon_8|\Omega||\partial\Omega|,$$

$$m_2 = \frac{\lambda_2\varrho}{2}\frac{l(\varepsilon)}{\varepsilon_2} + \frac{\lambda_2\delta}{2}\frac{l(\varepsilon)}{\varepsilon_3} + \frac{\lambda_2\delta}{2}\frac{l(\varepsilon)}{\varepsilon_4} + \frac{\lambda_2\gamma}{2}\left(\frac{l(\varepsilon)}{\varepsilon_5} + \varepsilon_5|\Omega||\partial\Omega|\right) + \frac{\lambda_2\gamma}{2}\frac{l(\varepsilon)}{\varepsilon_6} + \frac{\lambda_3\gamma}{2}\varepsilon_{10}|\Omega||\partial\Omega| + \frac{\lambda_1}{2},$$

$$m_3 = \frac{\lambda_3\varrho}{2}\frac{l(\varepsilon)}{\varepsilon_7} + \frac{\lambda_3\delta}{2}\frac{l(\varepsilon)}{\varepsilon_8} + \frac{\lambda_3\delta}{2}\frac{l(\varepsilon)}{\varepsilon_9} + \frac{\lambda_3\gamma}{2}\frac{l(\varepsilon)}{\varepsilon_{10}} + \frac{\lambda_3\gamma}{2}\frac{l(\varepsilon)}{\varepsilon_{11}} + \frac{\lambda_2}{2} + \frac{\lambda_1\alpha}{2},$$

$$m_4 = \frac{\lambda_1\varrho}{2\varepsilon_1}\varepsilon + \frac{\lambda_2\varrho}{2}\varepsilon_{12} + \frac{\lambda_2\varrho}{2}\varepsilon_{13},$$

$$m_5 = \frac{\lambda_2\varrho}{2}\frac{\varepsilon}{\varepsilon_2} + \frac{\lambda_2\delta}{2}\frac{\varepsilon}{\varepsilon_3} + \frac{\lambda_2\delta}{2}\frac{\varepsilon}{\varepsilon_4} + \frac{\lambda_2\gamma}{2}\frac{\varepsilon}{\varepsilon_5} + \frac{\lambda_2\gamma}{2}\frac{\varepsilon}{\varepsilon_6} + \frac{\lambda_2\varrho}{2\varepsilon_{12}} + \frac{\lambda_3\delta\varepsilon_{14}}{2}, \tag{62}$$

we have

$$\begin{aligned}
&\frac{\lambda_1\varrho}{2\varepsilon_1}l(\varepsilon)\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)} + \frac{\lambda_1\beta}{2}\left\|u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} + \left(\frac{\lambda_2\alpha}{2} + \frac{\lambda_3\beta}{2}\right)\left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} \\
&+\left\{\frac{\lambda_3}{2} - \frac{\lambda_1}{2} - \frac{\lambda_2}{2}\right\}\left\|u^N_{\tau\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} + \frac{\lambda_1\varrho}{2}\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)} \\
&+\left\{\frac{\lambda_1\gamma}{2} + \frac{\lambda_2\delta}{2}\right\}\left\|\nabla u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} \\
&+\left\{-\frac{\lambda_3\varrho}{2}\frac{\varepsilon}{\varepsilon_7} - \frac{\lambda_3\delta}{2}\frac{\varepsilon}{\varepsilon_8} - \frac{\lambda_3\delta}{2}\frac{\varepsilon}{\varepsilon_9} - \frac{\lambda_3\gamma}{2}\frac{\varepsilon}{\varepsilon_{10}} - \frac{\lambda_3\gamma}{2}\frac{\varepsilon}{\varepsilon_{11}} - \frac{\lambda_2\varrho}{2\varepsilon_{13}} - \frac{\lambda_3\delta}{2\varepsilon_{14}} + \frac{\lambda_3\gamma}{2}\right\} \cdot \left\|\nabla u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} \\
&\le\left\{\frac{\lambda_1\delta}{2}|\Omega||\partial\Omega|T + \frac{\lambda_2\delta}{2}\varepsilon_4|\Omega||\partial\Omega| + \frac{\lambda_3\delta}{2}\varepsilon_9|\Omega||\partial\Omega| + m_1\right\} \cdot \left\|u^N(x,0)\right\|^2_{L^2(\Omega)} \\
&\left\{\frac{\lambda_1\gamma}{2}|\Omega||\partial\Omega|T + \frac{\lambda_2\gamma}{2}\varepsilon_6|\Omega||\partial\Omega| + \frac{\lambda_3\gamma}{2}\varepsilon_{11}|\Omega||\partial\Omega| + \frac{\lambda_1}{2} + \frac{\lambda_1\alpha}{2} + \frac{\lambda_1\beta}{2} + m_2\right\} \cdot \left\|u^N_t(x,0)\right\|^2_{L^2(\Omega)} \\
&+\left\{\frac{\lambda_2}{2} + \frac{\lambda_1\alpha}{2} + \left(\frac{\lambda_2\alpha}{2} - \frac{\lambda_3\beta}{2}\right) + m_3\right\}\left\|u^N_{tt}(x,0)\right\|^2_{L^2(\Omega)} + \left\{\frac{\lambda_1}{2} + \frac{\lambda_2}{2} + \frac{\lambda_3}{2}\right\}\left\|u^N_{ttt}(x,0)\right\|^2_{L^2(\Omega)} \\
&+\left\{\frac{\lambda_1\varrho}{2} + \frac{\lambda_2\varrho}{2} + \frac{\lambda_3\varrho}{2} + m_4\right\}\left\|\nabla u^N(x,0)\right\|^2_{L^2(\Omega)} + \left\{\frac{\lambda_2\varrho}{2} + \frac{\lambda_3\delta}{2} + \frac{\lambda_1\gamma}{2} + \frac{\lambda_2\delta}{2} + m_5\right\}\left\|\nabla u^N_t(x,0)\right\|^2_{L^2(\Omega)}, \\
&+\left\{\frac{\lambda_3\varrho}{2} + \frac{\lambda_3\delta}{2} - \frac{\lambda_3\gamma}{2}\right\}\left\|\nabla u^N_{tt}(x,0)\right\|^2_{L^2(\Omega)} + (\gamma_1 + m_1)\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+(\gamma_2 + m_1 + m_2)\int_0^\tau\left\|u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt + (\gamma_3 + m_2 + m_3)\int_0^\tau \cdot \left\|u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+\left\{\frac{\lambda_1}{2} + \lambda_2 - \lambda_3\alpha + m_3\right\}\int_0^\tau\left\|u^N_{ttt}(x,t)\right\|^2_{L^2(\Omega)}dt + \left\{\frac{\lambda_1\varrho}{2}\varepsilon + m_4\right\}\int_0^\tau\left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+(\gamma_4 + m_4 + m_5)\int_0^\tau\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt + (\gamma_5 + m_5)\int_0^\tau\left\|\nabla u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}dt,
\end{aligned} \tag{63}$$

where

$$\gamma_1 = \frac{\lambda_1\varrho}{2}\varepsilon_1 T|\Omega||\partial\Omega| + \frac{\lambda_1\varrho}{2}(l(\varepsilon) + |\Omega||\partial\Omega|) + \left(\frac{\lambda_1\delta + \lambda_2\varrho}{2}\right)|\Omega||\partial\Omega| + \frac{\lambda_2\varrho}{2}\varepsilon_2 T|\Omega||\partial\Omega| + \frac{\lambda_3\varrho}{2}\varepsilon_7 T|\Omega||\partial\Omega| + \frac{\lambda_3\varrho}{2}|\Omega||\partial\Omega|,$$

$$\gamma_2 = \left(\frac{\lambda_1\delta + \lambda_2\varrho}{2}\right)l(\varepsilon) + \frac{\lambda_1\delta}{2}l(\varepsilon) + \left(\frac{\lambda_1\gamma + \lambda_2\delta}{2}\right)(l(\varepsilon) + |\Omega||\partial\Omega|) + \frac{\lambda_1\gamma}{2}l(\varepsilon) + \frac{\lambda_2\gamma}{2}l(\varepsilon) + \frac{\lambda_3\delta}{2}|\Omega||\partial\Omega|,$$

$$\gamma_3 = \frac{\lambda_2\gamma}{2}|\Omega||\partial\Omega| + \frac{\lambda_3\varrho}{2}l(\varepsilon) + \frac{\lambda_3\delta}{2}l(\varepsilon) + \frac{\lambda_3\gamma}{2}(l(\varepsilon) + |\Omega||\partial\Omega|) + \frac{\lambda_1}{2} + (\lambda_1\alpha - \lambda_2\beta),$$

$$\gamma_4 = \left(\frac{\lambda_1\delta + \lambda_2\varrho}{2}\right)\varepsilon + \frac{\lambda_1\delta}{2}\varepsilon + \left(\frac{\lambda_1\gamma + \lambda_2\delta}{2}\right)\varepsilon + \frac{\lambda_1\gamma}{2}\varepsilon + \frac{\lambda_2\gamma}{2}\varepsilon + \frac{\lambda_3\varrho}{2} + (\lambda_2\rho - \lambda_1\delta),$$

$$\gamma_5 = \frac{\lambda_3\delta}{2}\varepsilon + \frac{\lambda_3\gamma}{2}\varepsilon + \frac{\lambda_3\varrho}{2} + (\lambda_3\delta - \lambda_2\gamma). \tag{64}$$

Choosing $\varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10}, \varepsilon_{11}, \varepsilon_{113}$, and ε_{14} sufficiently large

$$\frac{\lambda_3\varrho}{2}\frac{\varepsilon}{\varepsilon_7}+\frac{\lambda_3\delta}{2}\frac{\varepsilon}{\varepsilon_8}+\frac{\lambda_3\delta}{2}\frac{\varepsilon}{\varepsilon_9}+\frac{\lambda_3\gamma}{2}\frac{\varepsilon}{\varepsilon_{10}}+\frac{\lambda_3\gamma}{2}\frac{\varepsilon}{\varepsilon_{11}}+\frac{\lambda_3\delta}{2\varepsilon_{14}}+\frac{\lambda_2\varrho}{2\varepsilon_{13}}<\frac{\lambda_3\gamma}{2}, \tag{65}$$

the relation (80) reduces to

$$\begin{aligned}
&\Big\{\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)}\\
&\quad+\left\|\nabla u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)}\\
&+\left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|u^N_{\tau\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}\Big\}\\
&\le D\int_0^\tau\Big\{\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)}+\left\|u^N_t(x,t)\right\|^2_{L^2(\Omega)}\\
&\quad+\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}+\left\|u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}\\
&\quad+\left\|u^N_{ttt}(x,t)\right\|^2_{L^2(\Omega)}\Big\}dt\\
&\quad+D\Big\{\left\|u^N(x,0)\right\|^2_{W_2^1(\Omega)}+\left\|u^N_t(x,0)\right\|^2_{W_2^1(\Omega)}\\
&\quad+\left\|u^N_{tt}(x,0)\right\|^2_{W_2^1(\Omega)}+\left\|u^N_{ttt}(x,0)\right\|^2_{L^2(\Omega)}\Big\},
\end{aligned}\tag{66}$$

where

$$D:=\frac{\begin{gathered}\max\{(\lambda_1\delta/2)|\Omega||\partial\Omega|T+(\lambda_2\delta/2)\varepsilon_4|\Omega||\partial\Omega|+(\lambda_3\delta/2)\varepsilon_9|\Omega||\partial\Omega|+m_1,(\lambda_1\gamma/2)|\Omega||\partial\Omega|T+(\lambda_2\gamma/2)\varepsilon_6|\Omega||\partial\Omega|\\+(\lambda_3\gamma/2)\varepsilon_{11}|\Omega||\partial\Omega|+(\lambda_1/2)+(\lambda_1\alpha/2)+(\lambda_1\beta/2)+m_2,(\lambda_2/2)+(\lambda_1\alpha/2)+(\lambda_2\alpha/2)-(\lambda_3\beta/2)+m_3,(\lambda_1/2)+(\lambda_2/2)+(\lambda_3/2)\\(\lambda_1\varrho/2)+(\lambda_2\varrho/2)+(\lambda_3\varrho/2)+m_4,(\lambda_2\varrho/2)+(\lambda_3\delta/2)+(\lambda_1\gamma/2)+(\lambda_2\delta/2)+m_5,(\lambda_3\varrho/2)+(\lambda_3\delta/2)-\lambda_3\gamma/2,\\\gamma_1+m_1,\gamma_2+m_1+m_2,\gamma_3+m_2+m_3,(\lambda_1/2)+\lambda_2-\lambda_3\alpha+m_3,\\(\lambda_1\varrho/2)\varepsilon+m_4,\gamma_4+m_4+m_5,\gamma_5+m_5\}\end{gathered}}{\begin{gathered}\min\{(\lambda_1\varrho/2\varepsilon_1)l(\varepsilon),(\lambda_1\beta/2),(\lambda_2\alpha/2)+(\lambda_3\beta/2),\\\lambda_3/2-\lambda_1/2-\lambda_2/2,\lambda_1\varrho/2,(\lambda_1\gamma/2)+(\lambda_2\delta/2),\\,-(\lambda_3\varrho/2)(\varepsilon/\varepsilon_7)-(\lambda_3\delta/2)(\varepsilon/\varepsilon_8)-(\lambda_3\delta/2)(\varepsilon/\varepsilon_9)-(\lambda_3\gamma/2)(\varepsilon/\varepsilon_{10})-(\lambda_3\gamma/2)(\varepsilon/\varepsilon_{11})-\lambda_2\varrho/2\varepsilon_{13}-\lambda_3\delta/2\varepsilon_{14}-\lambda_3\gamma/2\}\end{gathered}}. \tag{67}$$

Applying the Gronwall inequality to (60) and then integrating from 0 to τ, it appears that

$$\left\{\begin{aligned}&\left\|u^N(x,t)\right\|^2_{W_2^1(Q_\tau)}+\left\|u^N_t(x,t)\right\|^2_{W_2^1(Q_\tau)}+\left\|u^N_{tt}(x,t)\right\|^2_{W_2^1(Q_\tau)}\\&\le De^{DT}\Big\{\|u_0(x)\|^2_{W_2^1(\Omega)}+\|u_1(x)\|^2_{W_2^1(\Omega)}+\|u_2(x)\|^2_{L^2(\Omega)}+\|u_3(x)\|^2_{L^2(\Omega)}\Big\}.\end{aligned}\right. \tag{68}$$

We deduce from (84) that

$$\left\|u^N(x,t)\right\|^2_{w_2^1(Q_\tau)}+\left\|u^N_t(x,t)\right\|^2_{w_2^1(Q_\tau)}+\left\|u^N_{tt}(x,t)\right\|^2_{w_2^1(Q_\tau)}\le A. \tag{69}$$

Therefore, the sequence $\{u^N\}_{N\ge1}$ is bounded in $V(Q_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(Q_T)$ to a limit function $u(x,t)$; we have to show that $u(x,t)$ is a generalized solution of (4). Since $u^N(x,t)\to u(x,t)$ in $L^2(Q_T)$ and $u^N(x,0)\to\zeta(x)$ in $L^2(\Omega)$, then $u(x,0)=\zeta(x)$.

Now to prove that (3) holds, we multiply each relation in (15) by a function $p_l(t)\in W_2^1(0,T)$, $p_l(t)=0$, then add up the obtained equalities ranging from $l=1$ to $l=N$, and integrate over t on $(0,T)$. If we let $\eta^N=\sum_{k=1}^N p_k(t)Z_k(x)$, then we have

$$\begin{aligned}
&-\left(u^N_{ttt},\eta^N_t\right)_{L^2(Q_T)}-\alpha\left(u^N_{tt},\eta^N_t\right)_{L^2(Q_T)}-\beta\left(u^N_t,\eta^N_t\right)_{L^2(Q_T)}\\
&+\varrho\left(\nabla u^N,\nabla\eta^N\right)_{L^2(Q_T)}+\delta\left(\nabla u^N_t,\nabla\eta^N\right)_{L^2(Q_T)}-\gamma\left(\nabla u^N_t,\nabla\eta^N_t\right)_{L^2(Q_T)}\\
&=\varrho\int_{\partial\Omega}\int_0^T\eta^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\tau)d\xi d\tau\right)dtds_x\\
&\quad+\delta\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_\Omega u^N(\xi,t)d\xi dtds_x\\
&\quad-\delta\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_\Omega u^N(\xi,0)d\xi dtds_x\\
&\quad-\gamma\int_0^T\int_{\partial\Omega}\eta^N_t\left(\int_\Omega u^N(\xi,t)d\xi\right)ds_xdt\\
&\quad+\gamma\int_0^T\int_{\partial\Omega}\eta^N_t\left(\int_\Omega u^N(\xi,0)d\xi\right)ds_xdt\\
&\quad-\gamma\left(\Delta u^N_t(x,0),\eta^N(0)\right)_{L^2(\Omega)}
\end{aligned}$$

$$+\left(u_{ttt}^N(x,0),\eta^N(0)\right)_{L^2(\Omega)}+\alpha\left(u_{tt}^N(x,0),\eta^N(0)\right)_{L^2(\Omega)}+\beta\left(u_{tt}^N(x,0),\eta^N(0)\right)_{L^2(\Omega)}, \tag{70}$$

for all η^N of the form $\sum_{k=1}^N p_l(t)Z_k(x)$.

Since

$$\begin{aligned}
&\int_0^t\int_\Omega\left(\left(u^N(\xi,\tau)-u(\xi,\tau)\right)d\xi d\tau\le\sqrt{T|\Omega|}\left\|u^N-u\right\|_{L^2(Q_T)},\\
&\int_0^T\eta^N(x,t)\int_\Omega\left(u_t^N(\xi,t)-u_t(\xi,t)\right)d\xi dt\\
&\quad\le\sqrt{|\Omega|}\left(\int_0^T\left(\eta^N(x,t)\right)^2dt\right)^{1/2}\left\|u_t^N-u_t\right\|_{L^2(Q_T)},\\
&\int_0^T\eta^N(x,t)\int_\Omega\left(u(^N(\xi,0)-u(\xi,0)\right)d\xi dt\\
&\quad\le\sqrt{|\Omega|}\left(\int_0^T\left(\eta^N(x,t)\right)^2\right)^{1/2}\left\|u^N(x,0)\right\|_{L^2(Q_T)},\\
&\left\|u^N-u\right\|_{L^2(Q_T)}\to0,\ \text{as}N\to\infty,
\end{aligned}\tag{71}$$

therefore, we have

$$\begin{aligned}
&\varrho\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_0^t\int_\Omega u^N(\xi,\tau)d\xi d\tau dt ds_x,\\
&\to\varrho\int_{\partial\Omega}\int_0^T\eta(x,t)\int_0^t\int_\Omega u(\xi,\tau)d\xi d\tau dt ds_x,\\
&\delta\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x,\\
&\to\delta\int_{\partial\Omega}\int_0^T\eta(x,t)\int_\Omega u(\xi,t)d\xi dt ds_x,\\
&-\delta\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds,\\
&\to-\delta\int_{\partial\Omega}\int_0^T\eta(x,t)\int_\Omega u(\xi,0)d\xi dt ds,\\
&-\gamma\int_0^T\int_{\partial\Omega}\eta_t^N\left(\int_\Omega u^N(\xi,t)d\xi\right)ds_x dt,\\
&\to-\gamma\int_0^T\int_{\partial\Omega}\eta_t\left(\int_\Omega u(\xi,t)d\xi\right)ds_x dt,\\
&\gamma\int_0^T\int_{\partial\Omega}\eta_t^N\left(\int_\Omega u^N(\xi,0)d\xi\right)ds_x dt,\\
&\to\gamma\int_0^T\int_{\partial\Omega}\eta_t\left(\int_\Omega u(\xi,0)d\xi\right)ds_x dt.
\end{aligned}\tag{72}$$

Thus, the limit function u satisfies (3) for every $\eta^N=\sum_{k=1}^N p_l(t)Z_k(x)$. We denote by $\mathbb{Q}_N$ the totality of all functions of the form $\eta^N=\sum_{k=1}^N p_l(t)Z_k(x)$, with $p_l(t)\in W_2^1(0,T)$, $p_l(t)=0$.

But $\cup_{l=1}^N\mathbb{Q}_N$ is dense in $W(Q_T)$, and then relation (3) holds for all $u\in W(Q_T)$. Thus, we have shown that the limit function $u(x,t)$ is a generalized solution of problem (4) in $V(Q_T)$.

4. Uniqueness of Solution

Theorem 3. *The problem (4) cannot have more than one generalized solution in $V(Q_T)$.*

Proof. Suppose that there exist two different generalized solutions $u_1\in V(Q_T)$ and $u_2\in V(Q_T)$ for the problem (1). Then, the difference $U=u_1-u_2$ solves

$$\begin{cases}U_{tttt}+\alpha U_{ttt}+\beta U_{tt}-\varrho\Delta U-\delta\Delta U_t-\gamma\Delta U_{tt}=0,\\ U(x,0)=U_t(x,0)=U_{tt}(x,0)=U_{ttt}(x,0)=0\\ \dfrac{\partial u}{\partial\eta}=\displaystyle\int_0^t\int_\Omega u(\xi,\tau)d\xi d\tau,x\in\partial\Omega,\end{cases}\tag{73}$$

and (3) gives

$$\begin{aligned}
&-(U_{ttt},v_t)_{L^2(Q_T)}-\alpha(U_{tt},v_t)_{L^2(Q_T)}-\beta(U_t,v_t)_{L^2(Q_T)}\\
&+\varrho(\nabla U,\nabla v)_{L^2(Q_T)}+\delta(\nabla U_t,\nabla v)_{L^2(Q_T)}-\gamma(\nabla U_t,\nabla v_t)_{L^2(Q_T)}\\
&=\varrho\int_0^T\int_{\partial\Omega}v\left(\int_0^t\int_\Omega u(\xi,\tau)d\xi d\tau\right)ds_x dt\\
&+\delta\int_0^T\int_{\partial\Omega}v\int_\Omega u(\xi,t)d\xi ds_x dt-\gamma\int_0^T\int_{\partial\Omega}v_t\left(\int_\Omega u_\tau(\xi,t)d\xi dt\right)ds_x dt.
\end{aligned}\tag{74}$$

Consider the function

$$v(x,t)=\begin{cases}\displaystyle\int_t^\tau U(x,s)ds, & 0\le t\le\tau,\\ 0, & \tau\le t\le T.\end{cases}\tag{75}$$

It is obvious that $v\in W(Q_T)$ and $v_t(x,t)=-U(x,t)$ for all $t\in[0,\tau]$. Integration by parts in the left hand side of (75) gives

$$-(U_{ttt},v_t)_{L^2(Q_T)}=(U_{\tau\tau}(x,\tau),U(x,\tau))_{L^2(\Omega)}-\frac{1}{2}\|U_\tau(x,\tau)\|_{L^2(\Omega)}^2,\tag{76}$$

$$-\alpha(U_{tt},v_t)_{L^2(Q_T)}=\alpha(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)}-\alpha\int_0^\tau\|U_t(x,t)\|_{L^2(\Omega)}^2dt,\tag{77}$$

$$-\beta(U_t,v_t)_{L^2(Q_T)}=\frac{\beta}{2}\|U(x,\tau)\|_{L^2(\Omega)}^2,\tag{78}$$

$$\varrho(\nabla U,\nabla v)_{L^2(Q_T)} = \frac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)}, \quad (79)$$

$$\delta(\nabla U_t,\nabla v)_{L^2(Q_T)} = \delta\int_0^\tau \|\nabla v_t(x,t)\|^2_{L^2(\Omega)}dt, \quad (80)$$

$$-\gamma(\nabla U_t,\nabla v_t)_{L^2(Q_T)} = \frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}, \quad (81)$$

Plugging (76)–(95) into (88), we obtain

$$\begin{aligned}
&(U_{\tau\tau}(x,\tau),U(x,\tau))_{L^2(\Omega)} + \alpha(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)} \\
&\quad + \frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} + \frac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)} \\
&\quad + \frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&= \alpha\int_0^\tau \|U_t(x,t)\|^2_{L^2(\Omega)}dt - \delta\int_0^\tau \|\nabla v_t(x,t)\|^2_{L^2(\Omega)}dt \\
&\quad + \varrho\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)d\xi d\tau\right)ds_x dt \\
&\quad + \delta\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,t)d\xi ds_x dt \\
&\quad - \gamma\int_0^T\int_{\partial\Omega} v_t\left(\int_\Omega U(\xi,t)d\xi\right)ds dt.
\end{aligned} \quad (82)$$

Now, since

$$v^2(x,t) = \left(\int_t^\tau U(x,s)ds\right)^2 \le \tau\int_0^\tau U^2(x,s)ds, \quad (83)$$

then

$$\|v\|^2_{L^2(Q_\tau)} \le \tau^2\|U\|^2_{L^2(Q_\tau)} \le T^2\|U\|^2_{L^2(Q_\tau)}. \quad (84)$$

Using the trace inequality, the right-hand side of (96) can be estimated as follows:

$$\begin{aligned}
&\varrho\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)d\xi d\tau\right)ds_x dt \\
&\quad \le \frac{\varrho}{2}T^2\{l(\varepsilon)+|\Omega||\partial\Omega|\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt + \frac{\varrho}{2}\varepsilon\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt, \\
&\delta\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,t)d\xi ds_x dt \le \frac{\delta}{2}\left\{T^2 l(\varepsilon)+|\Omega||\partial\Omega|\right\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt \\
&\quad + \frac{\delta}{2}\varepsilon\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt, \\
&-\gamma\int_0^T\int_{\partial\Omega} v_t\left(\int_\Omega U(\xi,t)d\xi\right)ds dt = \gamma\int_0^T\int_{\partial\Omega} v\left(\int_\Omega U_t(\xi,t)d\xi\right)ds dt \\
&= \gamma\int_0^\tau\int_{\partial\Omega} v\left(\int_\Omega U_t(\xi,t)d\xi\right)ds dt \le \frac{\gamma|\Omega||\partial\Omega|}{2}\|U_t\|^2_{L^2(Q_\tau)} \\
&+ \frac{\eta^2}{2}\varepsilon\|\nabla v\|^2_{L^2(Q_\tau)} + \frac{\gamma}{2}l(\varepsilon)T^2\|U\|^2_{L^2(Q_\tau)}.
\end{aligned} \quad (85)$$

Combining the relations (98)-(101) and (96), we get

$$\begin{aligned}
&(U_{\tau\tau}(x,\tau),U(x,\tau))_{L^2(\Omega)} + \alpha(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)} \\
&\quad + \frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} \\
&+ \frac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&\le\left\{\frac{\varrho}{2}T^2(l(\varepsilon)+|\Omega||\partial\Omega|) + \frac{\delta}{2}\left(T^2 l(\varepsilon)+|\Omega||\partial\Omega|\right) + \frac{\gamma}{2}l(\varepsilon)T^2\right\} \\
&\quad \cdot\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left(\alpha + \frac{\gamma|\Omega||\partial\Omega|}{2}\right)\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left(\frac{\varrho+\delta+\gamma}{2}\right)\varepsilon\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt.
\end{aligned} \quad (86)$$

Next, multiplying the differential equation in (73) by U_{ttt} and integrating over $Q_\tau = \Omega\times(0,\tau)$, we obtain

$$\begin{aligned}
&(U_{tttt},U_{ttt})_{L^2(Q_\tau)} + \alpha(U_{ttt},U_{ttt})_{L^2(Q_\tau)} + \beta(U_{tt},U_{ttt})_{L^2(Q_\tau)} \\
&\quad - \varrho(\Delta U,U_{ttt})_{L^2(Q_\tau)} - \delta(\Delta U_t,U_{ttt})_{L^2(Q_\tau)} - \gamma(\Delta U_t,U_{ttt})_{L^2(Q_\tau)} = 0.
\end{aligned} \quad (87)$$

An integration by parts in (102) yields

$$(U_{tttt},U_{ttt})_{L^2(Q_\tau)} = \frac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}, \quad (88)$$

$$\alpha(U_{ttt},U_{ttt})_{L^2(Q_\tau)} = \alpha\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt, \quad (89)$$

$$\beta(U_{tt},U_{ttt})_{L^2(Q_\tau)} = \frac{\beta}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}, \quad (90)$$

$$\begin{aligned}
-\varrho(\Delta U,U_{ttt})_{L^2(Q_\tau)} &= \varrho\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau)_{L^2(\Omega)} \\
&\quad - \frac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \varrho\int_{\partial\Omega}U_{\tau\tau}(x,\tau) \\
&\quad \cdot\left(\int_0^\tau\int_\Omega U(\xi,\eta)d\xi d\eta\right)ds_x \\
&\quad + \varrho\int_{\partial\Omega}\int_0^\tau U_{tt}(x,t)\int_\Omega U(\xi,t)d\xi dt ds_x,
\end{aligned} \quad (91)$$

$$\begin{aligned}
-\delta(\Delta U_t,U_{ttt})_{L^2(Q_\tau)} &= \delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} \\
&\quad - \delta\int_0^\tau\|\nabla U_{tt}(x,)\|^2_{L^2(\Omega)}dt - \delta\int_{\partial\Omega}U_{\tau\tau}(x,\tau) \\
&\quad \cdot\int_\Omega U(\xi,\tau)d\xi ds_x + \delta\int_0^\tau\int_{\partial\Omega}U_{tt}(x,t) \\
&\quad \cdot\int_\Omega U_t(\xi,t)d\xi ds_x dt,
\end{aligned} \quad (92)$$

$$-\gamma(\Delta U_{tt}, U_{ttt})_{L^2(Q_\tau)} = \frac{\gamma}{2}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \gamma\int_{\partial\Omega} U_{\tau\tau}(x,\tau) \cdot \int_\Omega U_\tau(\xi,\tau)d\xi ds_x + \gamma\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t) \cdot \int_\Omega U_{tt}(\xi,t)d\xi ds_x dt. \quad (93)$$

Substituting (88)–(108) into (102), we get the equality

$$\begin{aligned}
&\frac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \varrho(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} \\
&+\delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} + \frac{\gamma}{2}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \frac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&= -\alpha\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt + \delta\int_0^\tau\|\nabla U_{tt}(x)\|^2_{L^2(\Omega)}dt \\
&+\varrho\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,\eta)d\xi d\eta\right)ds_x - \varrho\int_{\partial\Omega}\int_0^\tau U_{tt}(x,t)\int_\Omega U(\xi,t)d\xi dt ds_x \\
&+\delta\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U(\xi,\tau)d\xi ds_x - \delta\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_t(\xi,t)d\xi ds_x dt \\
&+\gamma\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U_\tau(\xi,\tau)d\xi ds_x - \gamma\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_{tt}(\xi,t)d\xi ds_x dt.
\end{aligned} \quad (94)$$

The right-hand side of (109) can be bounded as follows:

$$\begin{aligned}
&\varrho\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,\eta)d\xi d\eta\right)ds_x \\
&\quad\le \frac{\varrho}{2\varepsilon_1'}\left(\varepsilon\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\right) + \frac{\varrho}{2}\varepsilon_1' T|\partial\Omega||\Omega|\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}$$

$$\begin{aligned}
&-\varrho\int_{\partial\Omega}\int_0^\tau U_{tt}(x,t)\int_\Omega U(\xi,t)d\xi dt ds_x \\
&\quad\le \frac{\varrho}{2}\int_0^\tau\left\{\varepsilon\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{tt}(x,t)\|^2_{L^2(\Omega)}\right\}dt + \frac{\varrho}{2}|\Omega||\partial\Omega|\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}$$

$$\begin{aligned}
&\delta\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U(\xi,\tau)d\xi ds_x \\
&\quad\le \frac{\delta}{2\varepsilon_2'}\left(\varepsilon\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\right) + \frac{\delta}{2}\varepsilon_2' T|\Omega||\partial\Omega|\|U(x,\tau)\|^2_{L^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
&-\delta\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_t(\xi,t)d\xi ds_x dt \\
&\quad\le \frac{\delta}{2}\varepsilon\int_0^\tau\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt + \frac{\delta}{2}l(\varepsilon)\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt + \frac{\delta}{2}T|\Omega||\partial\Omega|\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}$$

$$\begin{aligned}
&\gamma\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U_\tau(\xi,\tau)d\xi ds_x \\
&\quad\le \frac{\gamma}{2\varepsilon_3'}\left(\varepsilon\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\right) + \frac{\gamma}{2}\varepsilon_3' T|\Omega||\partial\Omega|\|U_\tau(x,\tau)\|^2_{L^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
&-\gamma\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_{tt}(\xi,t)d\xi ds_x dt \\
&\quad\le \frac{\gamma}{2}l(\varepsilon)\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt + \frac{\gamma}{2}\varepsilon\int_0^\tau\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt + \frac{\gamma}{2}T|\Omega||\partial\Omega|\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt.
\end{aligned} \quad (95)$$

So, combining inequalities (110)–(115) and equality (109), we obtain

$$\begin{aligned}
&\frac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \left\{\frac{\beta}{2} - \frac{\varrho}{2\varepsilon_1'}l(\varepsilon) - \frac{\delta}{2\varepsilon_2'}l(\varepsilon) - \frac{\delta}{2}\varepsilon(\varepsilon)\right\}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&-\frac{\gamma}{2}\varepsilon_3' T|\Omega||\partial\Omega|\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \frac{\delta}{2}\varepsilon_2' T|\Omega||\partial\Omega|\|U(x,\tau)\|^2_{L^2(\Omega)} \\
&+\left\{\frac{\gamma}{2} - \frac{\varrho}{2\varepsilon_1'}\varepsilon - \frac{\delta}{2\varepsilon_2'}\varepsilon - \frac{\gamma}{2\varepsilon_3'}\varepsilon\right\}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \frac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&+\varrho(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} + \delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} \\
&\le -\alpha\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt + \left\{\frac{\varrho}{2}l(\varepsilon) + \frac{\delta}{2}l(\varepsilon) + \frac{\gamma}{2}l(\varepsilon) + \frac{\gamma}{2}T|\Omega||\partial\Omega|\right\} \cdot \int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\frac{\varrho}{2}\varepsilon_1' T|\partial\Omega||\Omega| + \frac{\varrho}{2}|\Omega||\partial\Omega|\right\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt + \frac{\delta}{2}T|\Omega||\partial\Omega|\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\delta + \frac{\varrho}{2}\varepsilon + \frac{\delta}{2}\varepsilon + \frac{\gamma}{2}\varepsilon\right\}\int_0^\tau\|\nabla U_{tt}(x)\|^2_{L^2(\Omega)}dt.
\end{aligned} \quad (96)$$

Adding side to side (101) and (116), we obtain

$$\begin{aligned}
&\left\{\frac{\beta}{2} - \frac{\delta}{2}\varepsilon_2' T|\Omega||\partial\Omega|\right\}\|U(x,\tau)\|^2_{L^2(\Omega)} + \left\{-\frac{1}{2} - \frac{\gamma}{2}\varepsilon_3' T|\Omega||\partial\Omega|\right\} \cdot \|U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&+\left\{\frac{\beta}{2} - \frac{\varrho}{2\varepsilon_1'}l(\varepsilon) - l(\varepsilon)\frac{\delta}{2\varepsilon_2'} - \frac{\gamma}{2\varepsilon_3'}l(\varepsilon)\right\}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&+(U_{\tau\tau}(x,\tau), U(x,\tau))_{L^2(\Omega)} + \alpha(U_\tau(x,\tau), U(x,\tau))_{L^2(\Omega)} + \frac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)}
\end{aligned}$$

$$+\varrho(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)}$$
$$+\delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)}$$

$$+\frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}-\frac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}$$
$$+\left\{\frac{\gamma}{2}-\frac{\varrho}{2\varepsilon_1'}\varepsilon-\frac{\delta}{2\varepsilon_2'}-\frac{\gamma}{2\varepsilon_3'}\varepsilon\right\}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}$$

$$\leq\left\{\frac{\varrho}{2}\varepsilon_1' T|\partial\Omega||\Omega|+\frac{\varrho}{2}|\Omega||\partial\Omega|+\frac{\varrho}{2}T^2(l(\varepsilon)+|\Omega||\partial\Omega|)\right.$$
$$+\frac{\delta}{2}\left(T^2 l(\varepsilon)+|\Omega||\partial\Omega|\right)$$

$$\left.+\frac{\gamma}{2}l(\varepsilon)T^2\right\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt$$

$$+\left(\alpha+\frac{\gamma|\Omega||\partial\Omega|}{2}+\frac{\delta}{2}T|\Omega||\partial\Omega|\right)\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt$$

$$+\left\{\frac{1}{2}+l(\varepsilon)\frac{\varrho}{2}+\frac{\delta}{2}l(\varepsilon)+\frac{\gamma}{2}l(\varepsilon)+\frac{\gamma}{2}T|\Omega||\partial\Omega|\right\}\int_0^\tau$$
$$\cdot\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt-\alpha\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt$$

$$+\left\{\frac{\delta}{2}\varepsilon+\frac{\gamma}{2}\varepsilon+\varepsilon\frac{\varrho}{2}+\delta\right\}\int_0^\tau\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt$$
$$+\left(\frac{\varrho+\delta+\gamma}{2}\right)\varepsilon\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt. \quad (97)$$

Now, to deal with the last term on the right-hand side of (117), we define the function $\theta(x,t)$ by the relation

$$\theta(x,t):=\int_0^t U(x,s)ds. \quad (98)$$

Hence, using (89), it follows that

$$v(x,t)=\theta(x,\tau)-\theta(x,t),\ \nabla v(x,0)=\nabla\theta(x,\tau),$$

$$\|\nabla v\|^2_{L^2(Q_\tau)}=\|\nabla\theta(x,\tau)-\nabla\theta(x,t)\|^2_{L^2(\Omega)}$$
$$\leq 2\left(\tau\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)}+\|\nabla\theta(x,t)\|^2_{L^2(Q_\tau)}\right). \quad (99)$$

And we make use of the following inequality:

$$-\frac{\alpha}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}-\frac{\alpha}{2}\|U(x,\tau)\|^2_{L^2(\Omega)}\leq\alpha(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)},$$
$$-\frac{1}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}\|U(x,\tau)\|^2_{L^2(\Omega)}\leq(U_{\tau\tau}(x,\tau),U(x,\tau))_{L^2(\Omega)},$$
$$-\frac{\varrho}{2\varepsilon_4'}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}-\frac{\varrho}{2}\varepsilon_4'\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}\leq\varrho(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)},$$
$$-\frac{\delta}{2\varepsilon_5'}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}-\frac{\delta}{2}\varepsilon_5'\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}\leq\delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)},$$
$$m_1\|U(x,\tau)\|^2_{L^2(\Omega)}\leq m_1\|U(x,t)\|^2_{L^2(Q_\tau)}+m_1\|U_t(x,t)\|^2_{L^2(Q_\tau)},$$
$$m_2\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}\leq m_2\|U_t(x,t)\|^2_{L^2(Q_\tau)}+m_2\|U_{tt}(x,t)\|^2_{L^2(Q_\tau)},$$
$$m_3\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\leq m_3\|U_{tt}(x,t)\|^2_{L^2(Q_\tau)}+m_3\|U_{ttt}(x,t)\|^2_{L^2(Q_\tau)},$$
$$m_4\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}\leq m_4\|\nabla U(x,t)\|^2_{L^2(Q_\tau)}+m_4\|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)},$$
$$m_5\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}\leq m_5\|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)}+m_5\|\nabla U_{tt}(x,t)\|^2_{L^2(Q_\tau)}. \quad (100)$$

Let

$$\begin{cases} m_1:=\frac{1}{2}+\frac{\delta}{2}\varepsilon_2' T|\Omega||\partial\Omega|+\frac{\alpha}{2} \\ m_2:=1+\frac{\gamma}{2}\varepsilon_3' T|\Omega||\partial\Omega|+\frac{\alpha}{2} \\ m_3:=\frac{\varrho}{2\varepsilon_1'}l(\varepsilon)+l(\varepsilon)\frac{\delta}{2\varepsilon_2'}+\frac{\gamma}{2\varepsilon_3'}l(\varepsilon)+\frac{1}{2} \\ m_4:=\frac{\varrho}{2}\varepsilon_4' \\ m_5:=1+\frac{\varrho}{2}+\frac{\delta}{2\varepsilon_5'}, \end{cases} \quad (101)$$

choosing ε_1', ε_2', ε_3', ε_4', and ε_5' sufficiently large:

$$\frac{\varrho}{2\varepsilon_1'}\varepsilon+\frac{\delta}{2\varepsilon_2'}+\frac{\gamma}{2\varepsilon_3'}\varepsilon+\frac{\varrho}{2\varepsilon_4'}+\frac{\delta}{2\varepsilon_5'}<\frac{\gamma}{2}. \quad (102)$$

Since τ is arbitrary, we get that $\varrho/2-\tau\varepsilon(\varrho+\delta+\gamma)>0$; thus, inequality (117) takes the form

$$\frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)}+\frac{1}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}+\frac{\beta}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}$$
$$+\frac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}$$

$$+\left\{\frac{\varrho}{2}-\tau\varepsilon(\varrho+\delta+\gamma)\right\}\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}$$
$$+\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}$$

$$+\left\{\frac{\gamma}{2}-\frac{\varrho}{2\varepsilon_1'}\varepsilon-\frac{\delta}{2\varepsilon_2'}-\frac{\gamma}{2\varepsilon_3'}\varepsilon-\frac{\varrho}{2\varepsilon_4'}-\frac{\delta}{2\varepsilon_5'}\right\}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}$$

$$\leq\left\{\gamma_1'+m_1\right\}\int_0^\tau\|U(x,t)\|_{L^2(\Omega)}^2dt+\left(\gamma_2'+m_1+m_2\right)\int_0^\tau\cdot\|U_t(x,t)\|_{L^2(\Omega)}^2dt$$
$$+\left\{\gamma_3'+m_2+m_3\right\}\int_0^\tau\|U_{tt}(x,t)\|_{L^2(\Omega)}^2dt+(m_3-\alpha)\int_0^\tau\cdot\|U_{ttt}(x,t)\|_{L^2(\Omega)}^2dt$$
$$+\varepsilon(\varrho+\delta+\gamma)\int_0^\tau\|\nabla\theta(x,t)\|_{L^2(\Omega)}^2dt$$
$$+m_4\int_0^\tau\|\nabla U(x,t)\|_{L^2(\Omega)}^2dt+(m_4+m_5)\int_0^\tau\|\nabla U_t(x,t)\|_{L^2(\Omega)}^2dt$$
$$+\left\{\frac{\delta}{2}\varepsilon+\frac{\gamma}{2}\varepsilon+\varepsilon\frac{\varrho}{2}+\delta+m_5\right\}\int_0^\tau\|\nabla U_{tt}(x,t)\|_{L^2(\Omega)}^2dt,\quad(103)$$

where

$$\begin{cases}\gamma_1':=\frac{\varrho}{2}\varepsilon_1'T|\partial\Omega||\Omega|+\frac{\varrho}{2}|\Omega||\partial\Omega|+\frac{\varrho}{2}T^2(l(\varepsilon)+|\Omega||\partial\Omega|)+\frac{\delta}{2}\left(T^2l(\varepsilon)+|\Omega||\partial\Omega|\right)+\frac{\gamma}{2}l(\varepsilon)T^2\\ \gamma_2':=\alpha+\frac{\gamma|\Omega||\partial\Omega|}{2}+\frac{\delta}{2}T|\Omega||\partial\Omega|\\ \gamma_3':=\frac{1}{2}+l(\varepsilon)\frac{\varrho}{2}+\frac{\delta}{2}l(\varepsilon)+\frac{\gamma}{2}l(\varepsilon)+\frac{\gamma}{2}T|\Omega||\partial\Omega|.\end{cases}\quad(104)$$

We obtain

$$\begin{aligned}&\|U(x,\tau)\|_{L^2(\Omega)}^2+\|U_\tau(x,\tau)\|_{L^2(\Omega)}^2+\|U_{\tau\tau}(x,\tau)\|_{L^2(\Omega)}^2+\|U_{\tau\tau\tau}(x,\tau)\|_{L^2(\Omega)}^2\\&+\|\nabla U(x,\tau)\|_{L^2(\Omega)}^2+\|\nabla U_\tau(x,\tau)\|_{L^2(\Omega)}^2+\|\nabla U_{\tau\tau}(x,\tau)\|_{L^2(\Omega)}^2\\&+\|\nabla\theta(x,\tau)\|_{L^2(\Omega)}^2\leq D\int_0^\tau\Big\{\|U(x,t)\|_{L^2(\Omega)}^2+\|U_t(x,t)\|_{L^2(\Omega)}^2+\|U_{tt}(x,t)\|_{L^2(\Omega)}^2\\&+\|U_{ttt}(x,t)\|_{L^2(\Omega)}^2+\|\nabla U(x,t)\|_{L^2(\Omega)}^2+\|\nabla U_t(x,t)\|_{L^2(\Omega)}^2+\|\nabla U_{tt}(x,t)\|_{L^2(\Omega)}^2\\&+\|\nabla\theta(x,t)\|_{L^2(\Omega)}^2\Big\}dt,\end{aligned}\quad(105)$$

where

$$D:=\frac{\max\left\{\left(\gamma_1'+m_1\right),\left(\gamma_2'+m_1+m_2\right)_4,\gamma_3'+m_2+m_3,m_3-\alpha,m_4,m_4+m_5,(\delta/2)\varepsilon+(\gamma/2)\varepsilon+\varepsilon(\varrho/2)+\delta+m_5,\varepsilon(\varrho+\delta+\gamma)\right\}}{\min\left\{(\beta/2),1/2,,(\gamma/2),\left\{\gamma/2-\left(\varrho/2\varepsilon_1'\right)\varepsilon-\delta/2\varepsilon_2'-\left(\gamma/2\varepsilon_3'\right)\varepsilon-\varrho/2\varepsilon_4'-\delta/2\varepsilon_5'\right\},,\{\varrho/2-\tau\varepsilon(\varrho+\delta+\gamma)\}\right\}}.\quad(106)$$

Further, applying Gronwall's lemma to (133), we deduce that

$$\begin{aligned}&\|U(x,\tau)\|_{L^2(\Omega)}^2+\|U_\tau(x,\tau)\|_{L^2(\Omega)}^2+\|U_{\tau\tau}(x,\tau)\|_{L^2(\Omega)}^2+\|U_{\tau\tau\tau}(x,\tau)\|_{L^2(\Omega)}^2\\&+\|\nabla U(x,\tau)\|_{L^2(\Omega)}^2+\|\nabla U_\tau(x,\tau)\|_{L^2(\Omega)}^2+\|\nabla U_{\tau\tau}(x,\tau)\|_{L^2(\Omega)}^2+\|\nabla\theta(x,\tau)\|_{L^2(\Omega)}^2\\&\leq0,\forall\tau\in\left[0,\frac{\varrho}{2\varepsilon(\rho+\delta+\gamma)}\right].\end{aligned}\quad(107)$$

We proceed in the same way for the intervals $\tau\in[((m-1)\varrho/2\varepsilon(\varrho+\delta+\gamma)),(m\varrho/2\varepsilon(\varrho+\delta+\gamma))]$ to cover the whole interval $[0,T]$, and thus proving that $U(x,\tau)=0$, for all τ in $[0,T]$. Thus, the uniqueness is proved.

5. Conclusion

In the study of the propagation of acoustic waves, it should be noted that the Moore–Gibson–Thompson equation is one of the equations of nonlinear acoustics describing acoustic wave propagation in gases and liquids. The behavior of acoustic waves depends strongly on the medium property related to dispersion, dissipation, and nonlinear effects. It arises from modeling high-frequency ultrasound (HFU) waves (see [10, 12, 34]). In this work, we have studied the solvability of the nonlocal mixed boundary value problem for the fourth order of the Moore–Gibson–Thompson equation. Galerkin's method was the main used tool for proving the solvability of the given nonlocal problem. In the next work, we will try to use the same method with the Hall-MHD equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see for example [45–48]) by using some famous algorithms (see [49–51])

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The fifth author extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research group program under grant (R.G.P.1/3/42).

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