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The Motion of a Test Particle around the Out-of-Plane Libration Points in the ER3BP with Oblateness up to Zonal Harmonics J_4

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Authors' contributions

*This work was carried out in collaboration between both authors. Author **JS** designed the study, managed the analyses of the study and managed the literature searches. Author **TKR** performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.*

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ABSTRACT

The study of the Elliptic Restricted Three-Body Problem (ER3BP) in our present work considered the $xz-plane$ coordinates taken oblateness up to zonal harmonics J_4 for Gliese 667 and Sirius systems. The $xz-plane$ coordinates considered here is termed; the out-of-plane libration points. The out-of-plane libration points is birthed from the existence of the three-dimsional Restricted Three-Body Problem (R3BP). Its points or positions are denoted by $L_{6,7}$. These positions $(L_{6,7})$ lie in the $xz-$plane almost directly above and below the center of each oblate primary. We have computed numerically the positions of the out-of-plane libration points $(L_{6,7})$ and its stability to show the effects of the parameters involved. With the help of the software MATHEMATICA, we have

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observed the topologies of Zero-Velocity Curves (ZVC) for the stated problem. It is found that, the positions of the out-of-plane libration points for the binary systems: Gliese 667 and Sirius seems to respectively move away and closer in the absence and present of oblateness. For the stability, it is evidenced that, for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane libration points are unstable for the binary systems mentioned above.

Keywords: Out-of-Plane; stability; libration points; ER3BP; Oblateness; Zonal Harmonics J_4.

1. INTRODUCTION

The motion of the general three-body problem is govern by eighteen first order, coupled and non-linear differential equations. However, only ten integrals of the motion are in existence; they are derived from the conservation of linear momentum, angular momentum and energy. Due to the nature of these equations of motion, solving them analytically becomes a challenge. It is by this that Lagrange assumed an infinitesimal mass (negligible mass) to be one of the masses possessed by the three-body problem. He called the three-body problem as the “Restricted Three-Body Problem (R3BP)”. The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around the origin of a mass on the premise of their mutual attraction and the negligible mass (infinitesimal) having no influence on the motion of the two finite masses. This type of R3BP is called the “Circular Restricted Three-Body Problem (CR3BP)”. The elliptic restricted three-body problem (ER3BP) describes the three-dimensional motion of a small particle, called the third body (infinitesimal mass) under the gravitational attraction force of two finite bodies, called the primaries, which revolve on elliptic orbits in a plane around their common center of mass. The motions of an asteroid, a space probe or an artificial satellite under the gravitational attraction of the Sun-Jupiter or Earth-Moon systems are typical examples.

The R3BP assumed the status of the most studied areas in space dynamics as well as celestial mechanics. Many results which are very important to our day to day activities have been produced by well-known mathematicians and scientists in a way to solve the problem involving the motion of natural bodies. Researches [1,2,3,4] among many have studied the collinear and non-collinear equilibrium points with varying parameters and chosen binary systems. Their results among many have shown a decrease in the size with an increase in the parameters involved when considering the stability nature of their studied problems.

SubbaRao [5], AbdulRaheem and Singh [6], Singh and Mohammed [7], Singh and Leke [8,9,10] and Singh and Tyokyaa [3] have investigated Restricted three-body problem when one or both primaries are taken as oblate spheroids in both circular and or elliptic form. Sharma and SubbaRao [11] considered the triangular libration points, taken the bigger primary as oblate spheroid whose equatorial plane agrees with the plane of motion. Their result asserted that, the oblateness of the primaries increase for both Coriolis and Centrifugal forces and the range of its linear stability at the triangular points decrease. This proves that the Coriolis force is not always a stabilizing force.

Numerous researches have affirmed the instability of collinear points in most cases [1,2,12,13,14]. Singh and Leke [8] studied the libration points and their stability in the R3BP with oblateness and variable masses. Their result shows that, the collinear points are stable due to k (kappa). However, it remains unstable in the out-of-plane libration points even with the introduction of k (kappa). Singh and Leke [10] in their study involving the binary star “post-AGB” and disc as a dust grain particle motion treatment around the non-collinear libration points affirmed that, the libration points around IRAS 11472-0800-G29-38 system are particularly unstable.

Javed and Shahbaz [15,16], considered the effects of Albedo and oblateness on the collinear and non-collinear libration points in the elliptic restricted problem of three-bodies. Recently, Javed et al. [17] studied the elliptic restricted three-body problem to show the effects of a dipole model over a synchronous system of Asteroids.

The equation of motion of the three-dimensional restricted three-body problem with oblateness of the primaries allow the existence of two families of the out-of-plane libration points denoted by $L_{6,7}$ and $L_{8,9}$. The first family $(L_{6,7})$, in the sun-planet-particle and Galaxy Kernel-sun-particle cases was first pointed out by [18]. These points lie in the xz −plane symmetrically with respect to the $x - axis$ along the curve almost directly above and below the center of each oblate primary. These points are denoted by $L_{6,7}$ [1,2,19,20,21].

Many studies have been carried out on the out-of-plane libration points. Among many, are [2,19,21,22,23]. Das et al. [19] considered the out-of-plane libration points $L_i(i = 6, 7, 8, 9)$ in an account of PR-drag of a passive micron size particle and their linear stability in the field of radiating binary systems (Kruger 60 and RW-monocerotis). They affirmed that, their positions are unstable in the presence of PR-drag and are stable without the introduction of PR-drag. Reference [1] studied the effects of Coriolis and centrifugal forces on the positions of the out-of-plane libration points when both primaries are radiating and oblating. An extension of [1] was carried out by [2]. Their study introduces two binary systems: Leporis and Altair. The Yarkovsky effects in a modified photogravitational three-bodies were observed by [23]. His study produces nine (9) points in the out-of-plane and about 256 of the out-of-plane libration points in existence. Singh and Amuda [21] considered the secondary primary as a source of radiation in their study of the PR-drag out-of-plane libration points in the circular photogravitational R3BP. Their results show instability in the out-of-plane libration points for the binary system Cen X-4. Our present work considered the $xz - plane$ coordinates in the ER3BP taken oblateness up to zonal harmonics J_4 of both primaries in the field of stellar binary systems; Gliese 667 and Sirius.

The paper is organized as follows: Sections 2 presents the equations of motion; Section 3 examine the positions of out-of-plane libration points; section 4 studied their stability; section 5 present the topologies of ZVC of the out-of-plane equilibrium points, section 6 explores numerical application and the discussions and conclusions are provided in section 7.

2. EQUATION OF MOTION

Consider a rotating frame of reference (o, x, y, z) with the origin at the centre of mass of the primaries (Szebehely [24]). The $x - axis$ lies along the line joining the two finite masses M_1 and M_2. The $y - axis$ is perpendicular to the $x - axis$ and lies in the plane of the orbits of the finite bodies. The $z - axis$ is perpendicular to the orbital plane of the finite bodies at the origin. Let r_1 and r_2 be the distances of the infinitesimal mass M_3 from the bigger and smaller primaries with masses M_1 at $(-x_1, 0, 0)$ and M_2 at $(x_2, 0, 0)$ respectively as shown in Fig. 1.

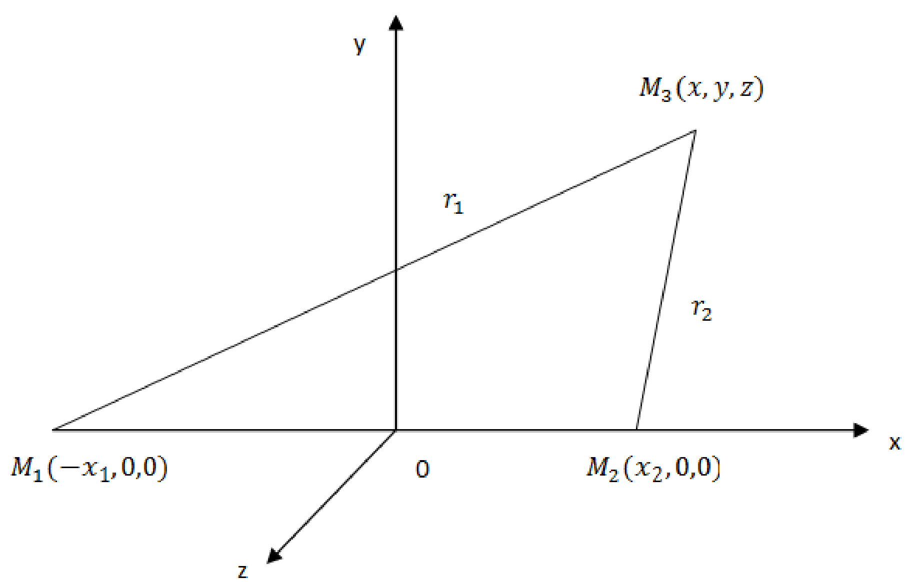


Fig. 1. Rotating frame of reference

We adopted the equation of motion under study from [2] and are presented here in dimensionless-pulsating coordinate system (x, y, z) as follows;

$$x^{''} - 2y^{'} = \frac{\delta\Omega}{\delta x}, y^{''} + 2x^{'} = \frac{\delta\Omega}{\delta y}, \qquad z^{''} = \frac{\delta\Omega}{\delta z} \tag{1}$$

with the force function

$$\Omega = (1 - e^2)^{-\frac{1}{2}}\left[\frac{1}{2}(x^2 + y^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1} + \frac{(1-\mu)A_1}{2{r_1}^3} - \frac{3(1-\mu)A_2}{8{r_1}^5} - \frac{3(1-\mu)A_1Z^2}{2{r_1}^5} + \frac{9(1-\mu)A_2Z^2}{8{r_1}^7} + \frac{\mu}{r_2} + \frac{\mu B_1}{2{r_2}^3} - \right.\right.$$

3μB28r25−3μB1Z22r25+9μB2Z28r27

(2)

The mean motion, n, is given as

$$n^2 = \frac{(1+e^2)^{\frac{1}{2}}}{a(1-e^2)}\left[1 + \frac{3}{2}(A_1 + B_1) - \frac{15}{8}(A_2 + B_2)\right] \tag{3}$$

$${r_i}^2 = (x - x_i)^2 + y^2 + z^2, (i = 1, 2) \quad x_1 = -\mu, \;\; x_2 = 1 - \mu\,, \;\; \mu = \frac{m_2}{m_1 + m_2} \tag{4}$$

Where, m_1, m_2 are the masses of the first (bigger) and second (smaller) primaries positioned at the points $(x_i, 0, 0), i = 1,2$; $A_i = J_{2i}{R_1}^2$ and $B_i = J_{2i}{R_2}^2$ $(i = 1,2)$ are the characterize zonal harmonic oblateness of the bigger and smaller primaries whose mean radii are R_1 and R_2 respectively. $\mu = \frac{m_2}{m_1+m_2}$ is the mass ratio, while a and e are the semi-major axis and eccentricity of the orbits, respectively.

2.1 The Potential of an Oblate Body

The potential of an oblate of mass M at a point distance r is given by

$$V = -\frac{MG}{r}\left\{1 - \frac{1}{r^2}J_2P_2 - \frac{1}{r^3}J_3P_3 - \frac{1}{r^4}J_4P_4 - \cdots\right\}$$

Where, P_n are the Legender Polynomial of order n $(n = 1, 2, \dots)$ and $J's$ are constants and are called zonal harmonics. The zonal harmonics are denoted by J_2 & J_4. For a spherical body, all the $J's$ will be zero. The inclusion of the odd $P's$ allow for a lack of symmetry about the equator (Danby [25]).

2.2 Ellipsoid

An ellipsoid is a closed quadric-surface that is a three-dimensional analogue of an ellipse

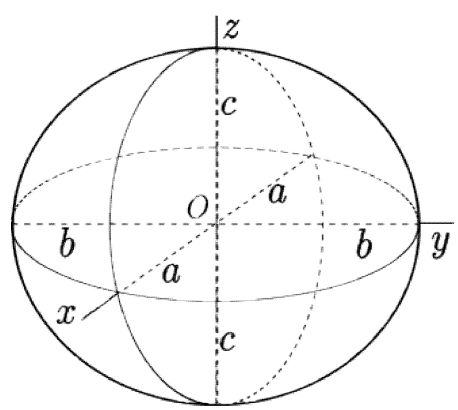


Fig. 2. Shape of an ellipsoid

In the above figure, if $a = b > c$, we have an oblate spheroid. In other words, an ellipsoid having a polar axis shorter than the diameter of the equatorial circle whose plane bisects is known as an oblate spheroid, thus, its two moments of inertial are equal out of the three. It is the approximated shape of many planets and celestial bodies, including Saturn, Jupiter and to a lesser extent the Earth. It is therefore the most used geometric figure for defining reference ellipsoid, upon which cartographic and geodetic system are based. The degree of flattening of a celestial object, such as a planet from a true spherical form, largely as a result of rotation is called oblateness.

3. POSITIONS OF OUT-OF-PLANE LIBRATION POINTS

To find the positions of the out-of-plane libration points denoted by $L_{6,7}$, we solve for the solutions of $\Omega_x = \Omega_y = \Omega_z = 0$ that is;

$$\Omega_x = (1-e^2)^{-\frac{1}{2}}\Big[x - \frac{1}{n^2}\Big(\frac{(1-\mu)(x+\mu)}{r_1{}^3} + \frac{3(1-\mu)(x+\mu)A_1}{2r_1{}^5} - \frac{15(1-\mu)(x+\mu)A_2}{8r_1{}^7} - \frac{15(1-\mu)(x+\mu)A_1Z^2}{2r_1{}^7} +$$

631−μx+μA2Z28r19+μx+μ−1r23+3μx+μ−1B12r25−15μx+μ−1B28r27−15μx+μ−1B1Z22r27 +63μx+μ−1B2Z28r29 (5)

$$\Omega_y = (1-e^2)^{-\frac{1}{2}}\Big[y\Big(1 - \frac{1}{n^2}\Big(\frac{(1-\mu)}{r_1{}^3} + \frac{3(1-\mu)A_1}{2r_1{}^5} - \frac{15(1-\mu)A_2}{8r_1{}^7} - \frac{15(1-\mu)A_1Z^2}{2r_1{}^7} + \frac{63(1-\mu)A_2Z^2}{8r_1{}^9} + \frac{\mu}{r_2{}^3} + \frac{3\mu B_1}{2r_2{}^5} -$$

15μB28r27−15μB1Z22r27+63μB2Z28r29
(6)

$$\Omega_z = \frac{(1-e^2)^{-\frac{1}{2}}}{n^2}\Big[-z\Big(\frac{(1-\mu)}{r_1{}^3} + \frac{3(1-\mu)A_1}{2r_1{}^5} - \frac{15(1-\mu)A_2}{8r_1{}^7} - \frac{15(1-\mu)A_1Z^2}{2r_1{}^7} + \frac{63(1-\mu)A_2Z^2}{8r_1{}^9} + \frac{\mu}{r_2{}^3} + \frac{3\mu B_1}{2r_2{}^5} - \frac{15\mu B_2}{8r_2{}^7} -$$

15μB1Z22r27+63μB2Z28r29
(7)

For the libration points, the solution of equations (5), (6) and (7) can be obtained by setting $\Omega_x = \Omega_y = \Omega_z = 0$

$$x - \frac{1}{n^2}\Big(\frac{(1-\mu)(x+\mu)}{r_1{}^3} + \frac{3(1-\mu)(x+\mu)A_1}{2r_1{}^5} - \frac{15(1-\mu)(x+\mu)A_2}{8r_1{}^7} - \frac{15(1-\mu)(x+\mu)A_1Z^2}{2r_1{}^7} + \frac{63(1-\mu)(x+\mu)A_2Z^2}{8r_1{}^9} + \frac{\mu(x+\mu-1)}{r_2{}^3} +$$

3μx+μ−1B12r25−15μx+μ−1B28r27−15μx+μ−1B1Z22r27+63μx+μ−1B2Z28r29=0
(8)

$$y\Big(1 - \frac{1}{n^2}\Big(\frac{(1-\mu)}{r_1{}^3} + \frac{3(1-\mu)A_1}{2r_1{}^5} - \frac{15(1-\mu)A_2}{8r_1{}^7} - \frac{15(1-\mu)A_1Z^2}{2r_1{}^7} + \frac{63(1-\mu)A_2Z^2}{8r_1{}^9} + \frac{\mu}{r_2{}^3} + \frac{3\mu B_1}{2r_2{}^5} - \frac{15\mu B_2}{8r_2{}^7} - \frac{15\mu B_1Z^2}{2r_2{}^7} +$$

63μB2Z28r29=0
(9)

$$-z\Big(\frac{(1-\mu)}{r_1{}^3} + \frac{3(1-\mu)A_1}{2r_1{}^5} - \frac{15(1-\mu)A_2}{8r_1{}^7} - \frac{15(1-\mu)A_1Z^2}{2r_1{}^7} + \frac{63(1-\mu)A_2Z^2}{8r_1{}^9} + \frac{\mu}{r_2{}^3} + \frac{3\mu B_1}{2r_2{}^5} - \frac{15\mu B_2}{8r_2{}^7} - \frac{15\mu B_1Z^2}{2r_2{}^7} +$$

63μB2Z28r29=0
(10)

Now, for the solutions of equations (8) and (10) with $y = 0$ and $z \neq 0$ equation (10) becomes

$$\frac{-(1-\mu)}{n^2r_1{}^3} - \frac{3(1-\mu)A_1}{2n^2r_1{}^5} + \frac{15(1-\mu)A_2}{8n^2r_1{}^7} + \frac{15(1-\mu)A_1Z^2}{2n^2r_1{}^7} - \frac{63(1-\mu)A_2Z^2}{8n^2r_1{}^9} - \frac{\mu}{n^2r_2{}^3} - \frac{3\mu B_1}{2n^2r_2{}^5} + \frac{15\mu B_2}{8n^2r_2{}^7} + \frac{15\mu B_1Z^2}{2n^2r_2{}^7} - \frac{63\mu B_2Z^2}{8n^2r_2{}^9} = 0 \qquad (11)$$

Multiplying equation (11) by $x - x_1$ and $x - x_2$ where $x_1 = -\mu$ and $x_2 = 1 - \mu$ we have respectively;

$$\frac{-(1-\mu)(x+\mu)}{n^2{r_1}^3}-\frac{3(1-\mu)(x+\mu)A_1}{2n^2{r_1}^5}+\frac{15(1-\mu)(x+\mu)A_2}{8n^2{r_1}^7}+\frac{15(1-\mu)(x+\mu)A_1Z^2}{2n^2{r_1}^7}-\frac{63(1-\mu)(x+\mu)A_2Z^2}{8n^2{r_1}^9}-\frac{\mu(x+\mu)}{n^2{r_2}^3}-\frac{3\mu(x+\mu)B_1}{2n^2{r_2}^5}+\frac{15\mu(x+\mu)B_2}{8n^2{r_2}^7}+\frac{15\mu(x+\mu)B_1Z^2}{2n^2{r_2}^7}-\frac{63\mu(x+\mu)B_2Z^2}{8n^2{r_2}^9}=0 \quad (12)$$

$$\frac{-(1-\mu)(x+\mu-1)}{n^2{r_1}^3}-\frac{3(1-\mu)(x+\mu-1)A_1}{2n^2{r_1}^5}+\frac{15(1-\mu)(x+\mu-1)A_2}{8n^2{r_1}^7}+\frac{15(1-\mu)(x+\mu-1)A_1Z^2}{2n^2{r_1}^7}-\frac{63(1-\mu)(x+\mu-1)A_2Z^2}{8n^2{r_1}^9}-\frac{\mu(x+\mu-1)}{n^2{r_2}^3}-\frac{3\mu(x+\mu-1)B_1}{2n^2{r_2}^5}+\frac{15\mu(x+\mu-1)B_2}{8n^2{r_2}^7}+\frac{15\mu(x+\mu-1)B_1Z^2}{2n^2{r_2}^7}-\frac{63\mu(x+\mu-1)B_2Z^2}{8n^2{r_2}^9}=0 \quad (13)$$

Subtracting equation (12) from (8) and substituting the value of $n^2=\frac{1}{a}\Big[1+\frac{3}{2}A_1+\frac{3}{2}B_1-\frac{15}{8}A_2-\frac{15}{8}B_2+$ *3e22* yields;

$$x=-\mu\left[1+\frac{3A_1}{2}+\frac{3B_1}{2}-\frac{15A_2}{8}-\frac{15B_2}{8}-\frac{3B_1}{2a^{2/3}}-\frac{15B_2}{8a^{4/3}}-\frac{15B_1Z^2}{2a^{4/3}}-\frac{63B_2Z^2}{2a^2}\right] \quad (14)$$

Now, from (4) we have

$$z^2=r_1^2-(x+\mu)^2,\ (i=1,2)\quad x_1=-\mu,\ \ x_2=1-\mu\,,\ \ \mu=\frac{m_2}{m_1+m_2} \quad (15)$$

But from Singh and Tyokyaa [3] we have;

$$r_1^2=a^{2/3}\left(1-e^2-A_1-B_1+\frac{5A_2}{4}+\frac{5B_2}{4}+A_1a^{-2/3}-\frac{5A_2a^{-4/3}}{4}\right),\qquad r_2^2=a^{2/3}\left(1-e^2-A_1-B_1+\frac{5A_2}{4}+\frac{5B_2}{4}+B_1a^{-2/3}-\frac{5B_2a^{-4/3}}{4}\right) \quad (16)$$

Considering equations (15) and (16) yields

$$z^2=\left[a^{2/3}\left(1-e^2-A_1-B_1+\frac{5A_2}{4}+\frac{5B_2}{4}+A_1a^{-2/3}-\frac{5A_2a^{-4/3}}{4}\right)-\mu^2(1+3A_1+3B_1-\frac{15A_2}{4}-\right.$$
15B24−3B1a−23+15B2a−434+15B1z2a−43−63B2z2a−24)

$$z=\left[a^{2/3}\left(1-e^2-A_1-B_1+\frac{5A_2}{4}+\frac{5B_2}{4}+A_1a^{-2/3}-\frac{5A_2a^{-4/3}}{4}\right)-\mu^2(1+3A_1+3B_1-\frac{15A_2}{4}-\frac{15B_2}{4}-\right.$$
3B1a−23+15B2a−434+15B1z2a−43−63B2z2a−24)12
(17)

Equations (14) and (17) present the location denoted by $L_{6,7}$ of the problem under review.

4. STABILITY OF OUT-OF-PLANE LIBRATION POINTS

To obtain the stability in the xz-plane of the restricted problem of three-bodies, we established the characteristics equation of the system under consideration.

Now, let the location of any of the libration point be denoted by (x_o, y_o, z_o) and suppose the small displacement of the location are (σ,β,α), then

$$x=x_o+\sigma,\ \ y=\ y_o \text{ and } \ z=z_o+\alpha$$

Taking derivatives, we have

$$x'=\sigma',x''=\sigma'',y'=\beta',y''=\beta'' \text{ and } z'=\alpha',z''=\alpha'' \quad (18)$$

Given the equations of motion of the infinitesimal mass by Singh and Tyokyaa [3] as;

$$x'' - 2y' = \frac{\partial\Omega}{\partial x}, \; y'' - 2x' = \frac{\partial\Omega}{\partial y} \text{ and } z'' = \frac{\partial\Omega}{\partial z} \tag{19}$$

We obtain the characteristics equation of the system as;

$$\lambda^6 + \left(4 - \Omega_{xx}^0 - \Omega_{yy}^0 - \Omega_{zz}^0\right)\lambda^4 + \left(\Omega_{xx}^0\Omega_{yy}^0 + \Omega_{yy}^0\Omega_{zz}^0 + \Omega_{xx}^0\Omega_{zz}^0 - 4\Omega_{zz}^0 - \left(\Omega_{xz}^0\right)^2\right)\lambda^2 - \left(\Omega_{xx}^0\Omega_{yy}^0\Omega_{zz}^0 - \left(\Omega_{xz}^0\right)^2\Omega_{yy}^0\right) = 0 \tag{20}$$

The superscripts o indicates that the partial derivatives are evaluated at the out-of-plane points under consideration. At the points under consideration ignoring products and higher order terms of very small parameters we have;

$$\Omega_{xx}^0 = (1-e^2)^{-1/2}\left[1 - \frac{3(1-\mu)}{4a^{2/3}} - \frac{3\mu}{4a^{2/3}} - \frac{3(1-\mu)A_1}{8a^{2/3}} + \frac{21\mu A_1}{8a^{2/3}} + \frac{21(1-\mu)B_1}{8a^{2/3}} - \frac{3\mu B_1}{8a^{2/3}} - \frac{3(1-\mu)e^2}{4a^{2/3}} - \frac{3\mu e^2}{4a^{2/3}} + \right.$$

1051−μA232a2+105μB232a2+1051−μA1z28a2+105μB1z28a2−451−μA232a23−45μA232a23 −451−μB232a23−45μB232a23 (21)

$$\Omega_{yy}^0 = (1-e^2)^{-1/2}\left[\frac{9(1-\mu)}{4a^{2/3}} + \frac{9\mu}{4a^{2/3}} - \frac{39(1-\mu)A_1}{8a^{2/3}} - \frac{39\mu A_1}{8a^{2/3}} - \frac{39(1-\mu)B_1}{8a^{2/3}} - \frac{39\mu B_1}{8a^{2/3}} - \frac{3(1-\mu)e^2}{4a^{2/3}} - \frac{3\mu e^2}{4a^{2/3}} + \right.$$

1951−μA232a23+195μA232a23+1951−μB232a23+195μB232a23−3151−μA232a2−315μB23 2a2−3151−μA1z28a2−315μB1z28a2 (22)

$$\Omega_{zz}^0 = (1-e^2)^{-1/2}\left[\frac{3\mu}{2a^{2/3}} - \frac{3(1-\mu)}{2a^{2/3}} + \frac{21(1-\mu)A_1}{4a^{2/3}} - \frac{9\mu A_1}{4a^{2/3}} + \frac{21(1-\mu)B_1}{4a^{2/3}} + \frac{15\mu B_1}{4a^{2/3}} - \frac{45(1-\mu)A_2}{16a^{2/3}} + \frac{45\mu A_2}{16a^{2/3}} + \right.$$

1051−μA216a2−451−μB216a23+45μB216a23−105μB216a2−31−μe22a23+3μe22a23+1051−μA1z24a2−105μB1z24a2 (23)

$$\Omega_{xz}^0 = (1-e^2)^{-1/2}\left[-\frac{3(1-\mu)}{8a^{2/3}} - \frac{3\mu}{8a^{2/3}} + \frac{51(1-\mu)A_1}{16a^{2/3}} + \frac{15\mu A_1}{16a^{2/3}} + \frac{63(1-\mu)B_1}{16a^{2/3}} - \frac{21\mu B_1}{16a^{2/3}} - \frac{45(1-\mu)A_2}{64a^{2/3}} - \frac{45\mu A_2}{64a^{2/3}} + \right.$$

1051−μA264a2−451−μB264a23−45μB264a23+105μB264a2−31−μe28a23−3μe28a23+1051−μA1z216a2+105μB1z216a2 (24)

Substituting equations (21)-(24) and neglecting higher order terms of A_1, B_1, A_2, B_2 & z with their products, we have;

$$\lambda^6 + P\lambda^4 + Q\lambda^2 - R = 0 \tag{25}$$

Where;

$$P = \frac{11}{2} + 2\mu + 3\mu\alpha + \left\{\frac{3}{2} - \frac{45\mu}{4}\right\}A_1 + \left\{-\frac{9}{4} + \frac{3\mu}{2}\right\}B_1 + \left\{-\frac{3}{4} + 4\mu\right\}e^2 + \left\{\frac{135}{16} + \frac{15\mu}{8}\right\}A_2 + \left\{\frac{75}{16} - \right.$$

225μ16B2+−1054+105μ4A1z2+315μ4B1z2 (26)

$$Q = \frac{27}{4} - 9\mu + \left\{\frac{9}{2} - 6\mu\right\}\alpha + \left\{-\frac{563}{32} + \frac{623\mu}{32}\right\}A_1 + \left\{-\frac{165}{8} + \frac{276\mu}{8}\right\}B_1 + \left\{\frac{15}{2} - 12\mu\right\}e^2 + \left\{-\frac{585}{32} + \right.$$

510μ32A2+13516−15μ16B2+−9458+945μ8A1z2+315μ8B1z2
(27)

$$Q = -\frac{81}{256} + \left\{-\frac{81}{128}\right\}\alpha + \left\{\frac{3{,}105}{512} - \frac{243\mu}{64}\right\}A_1 + \left\{\frac{3{,}753}{512} - \frac{4{,}536\mu}{512}\right\}B_1 + \left\{-\frac{513}{512} + \frac{540\mu^3}{256}\right\}e^2 + \left\{-\frac{4{,}185}{2{,}048}\right\}A_2 + \left\{-\frac{4{,}185}{2{,}048}\right\}B_2 \tag{28}$$

Now, equation (25) becomes;

$$\lambda^6 + \left[\frac{11}{2} + 2\mu + 3\mu\alpha + \left\{\frac{3}{2} - \frac{45\mu}{4}\right\}A_1 + \left\{-\frac{9}{4} + \frac{3\mu}{2}\right\}B_1 + \left\{-\frac{3}{4} + 4\mu\right\}e^2 + \left\{\frac{135}{16} + \frac{15\mu}{8}\right\}A_2 + \left\{\frac{75}{16} - \frac{225\mu}{16}\right\}B_2 + \left\{-\frac{105}{4} + \frac{105\mu}{4}\right\}A_1 z^2 + \left\{\frac{315\mu}{4}\right\}B_1 z^2\right]\lambda^4 + \left[\frac{27}{4} - 9\mu + \left\{\frac{9}{2} - 6\mu\right\}\alpha + \left\{-\frac{563}{32} + \frac{623\mu}{32}\right\}A_1 + \left\{-\frac{165}{8} + \frac{276\mu}{8}\right\}B_1 + \left\{\frac{15}{2} - 12\mu\right\}e^2 + \left\{-\frac{585}{32} + \frac{510\mu}{32}\right\}A_2 + \left\{\frac{135}{16} - \frac{15\mu}{16}\right\}B_2 + \left\{-\frac{945}{8} + \right.$$

$$\left.\frac{945\mu}{8}\right\} A_1 z^2 + \left\{\frac{315\mu}{8}\right\} B_1 z^2 \Big] \lambda^2 - \Big[-\frac{81}{256} + \left\{-\frac{81}{128}\right\} \alpha + \left\{\frac{3{,}105}{512} - \frac{243\mu}{64}\right\} A_1 + \left\{\frac{3{,}753}{512} - \frac{4{,}536\mu}{512}\right\} B_1 + \left\{-\frac{513}{512} + \frac{540\mu^3}{256}\right\} e^2 + \left\{-\frac{4{,}185}{2{,}048}\right\} A_2 + \left\{-\frac{4{,}185}{2{,}048}\right\} B_2 \Big] = 0 \quad (29)$$

5. ZERO-VELOCITY CURVES (ZVC) IN THE x, z- PLANE

Using equations (2), (3) and (4), we have presented our quantitative method as the premise to which the information about the motion of the third body near the out-of-plane libration points is understood. These Zero-Velocity Curves (ZVC) in the (x, z) are demonstrated considering the binary systems: Gliese 667 and Sirius. The effects of the parameters under study on the positions of the out-of-plane libration points are shown in Figs. 3 & 4 and Figs. 5 & 6 for Gliese 667 and Sirius systems respectively. These parameters have significant effects on the positions and stability of the problem under consideration for the aforementioned binary systems. The possible topologies of these curves are demonstrated in Figs. 3-6 for Gliese 667 and Sirius systems.

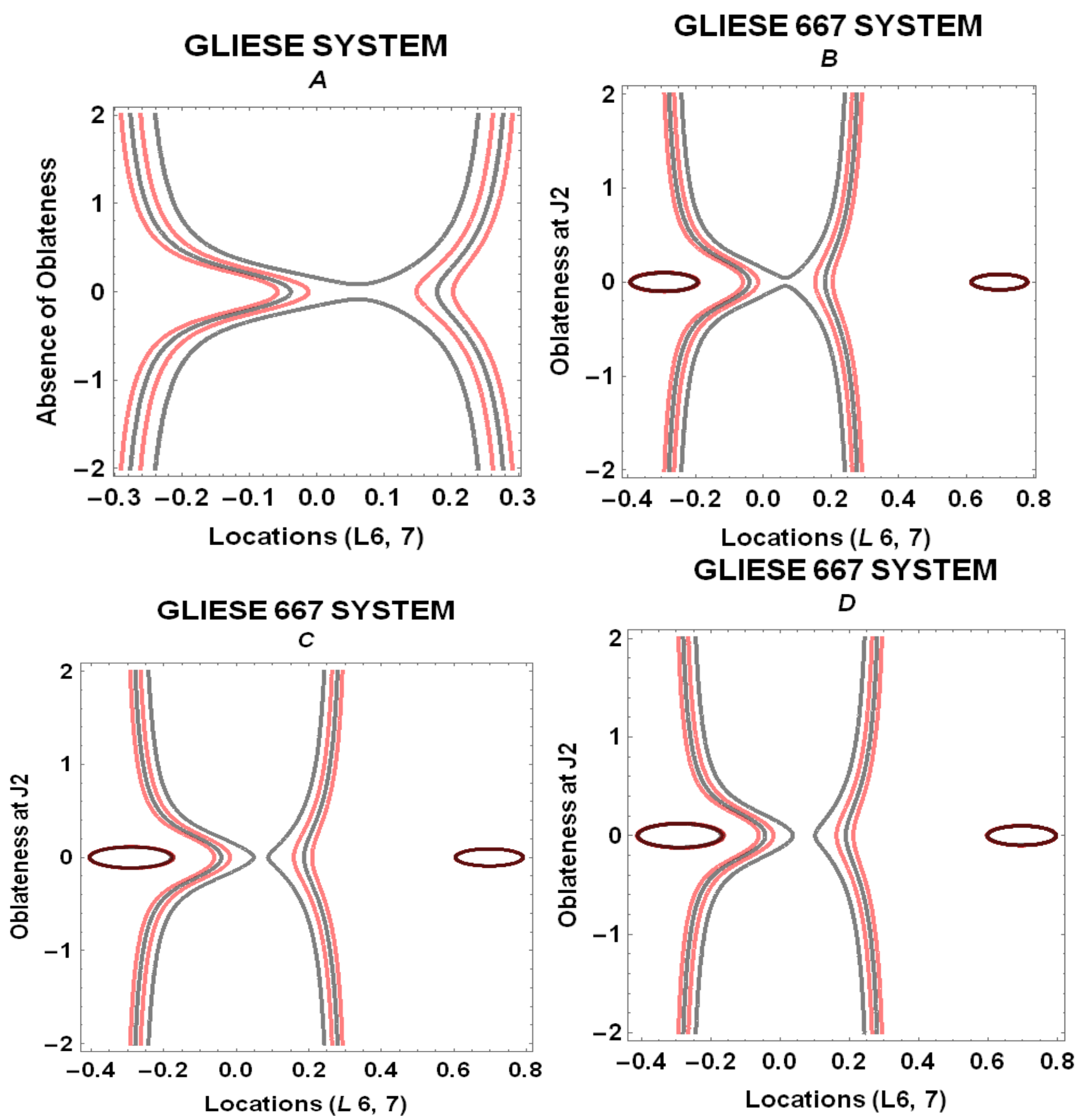


Fig. 3. Zero-Velocity Curves for Gliese 667 system with Oblateness at J_2 (A_1 & B_1)

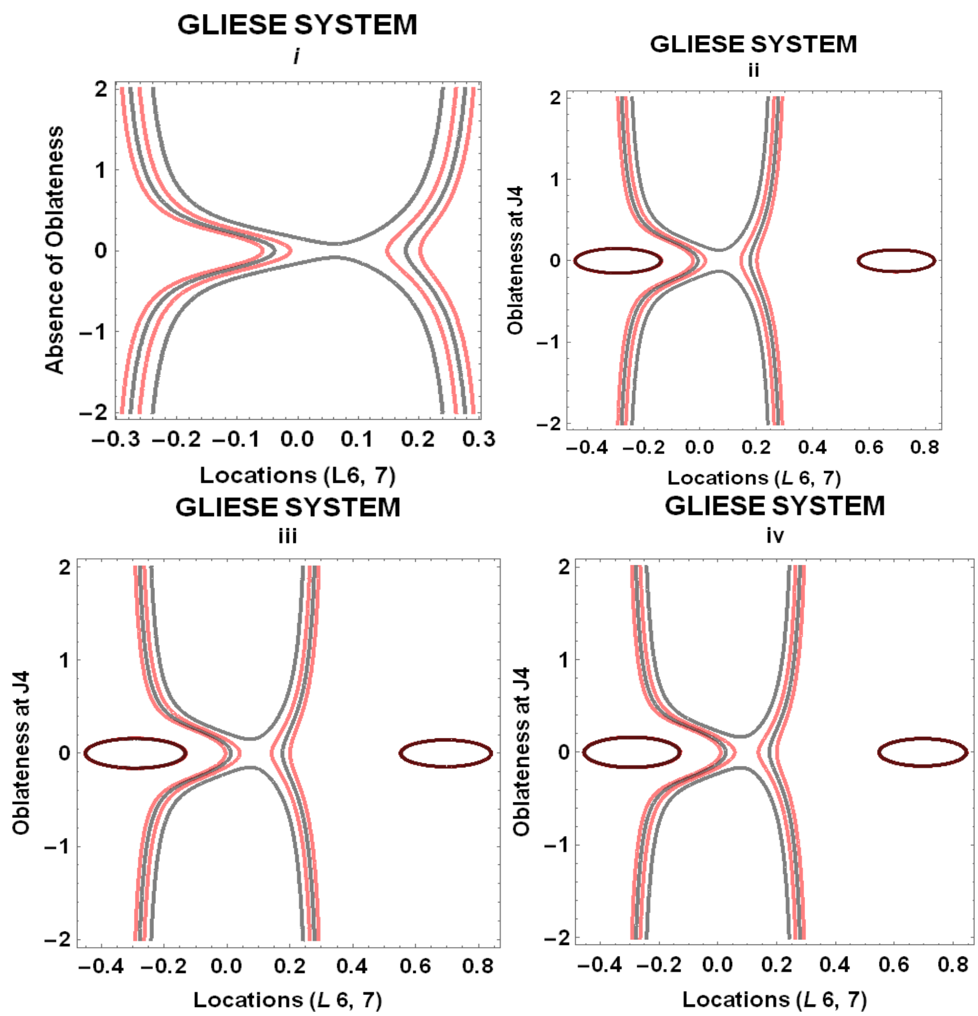
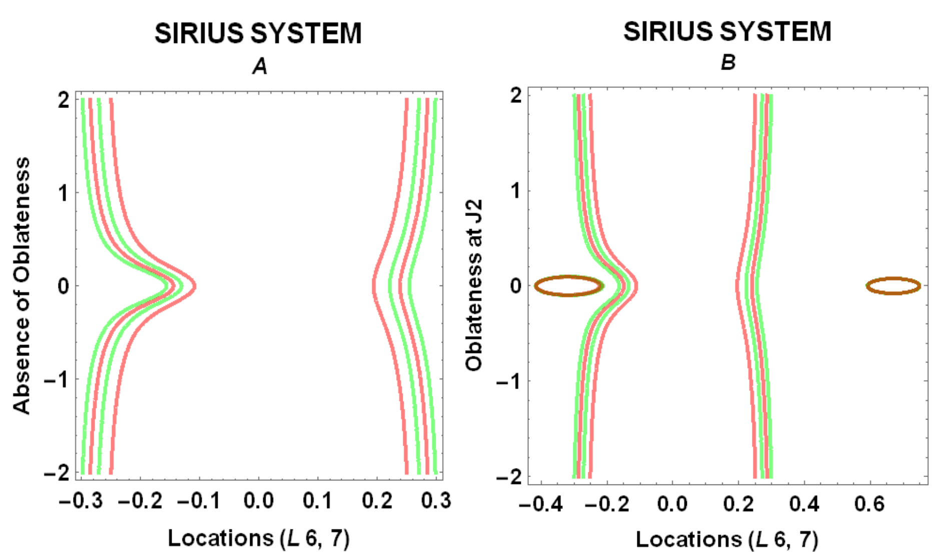


Fig. 4. Zero-Velocity Curves for Gliese 667 system with Oblateness at J_4 (A_2 & B_2)



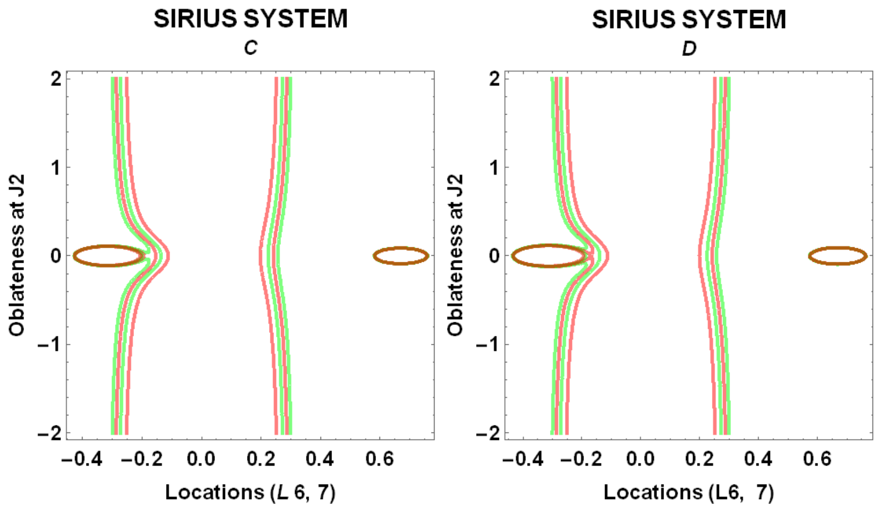


Fig. 5. Zero-Velocity Curves for Sirius system with Oblateness at J_2 (A_1 & B_1)

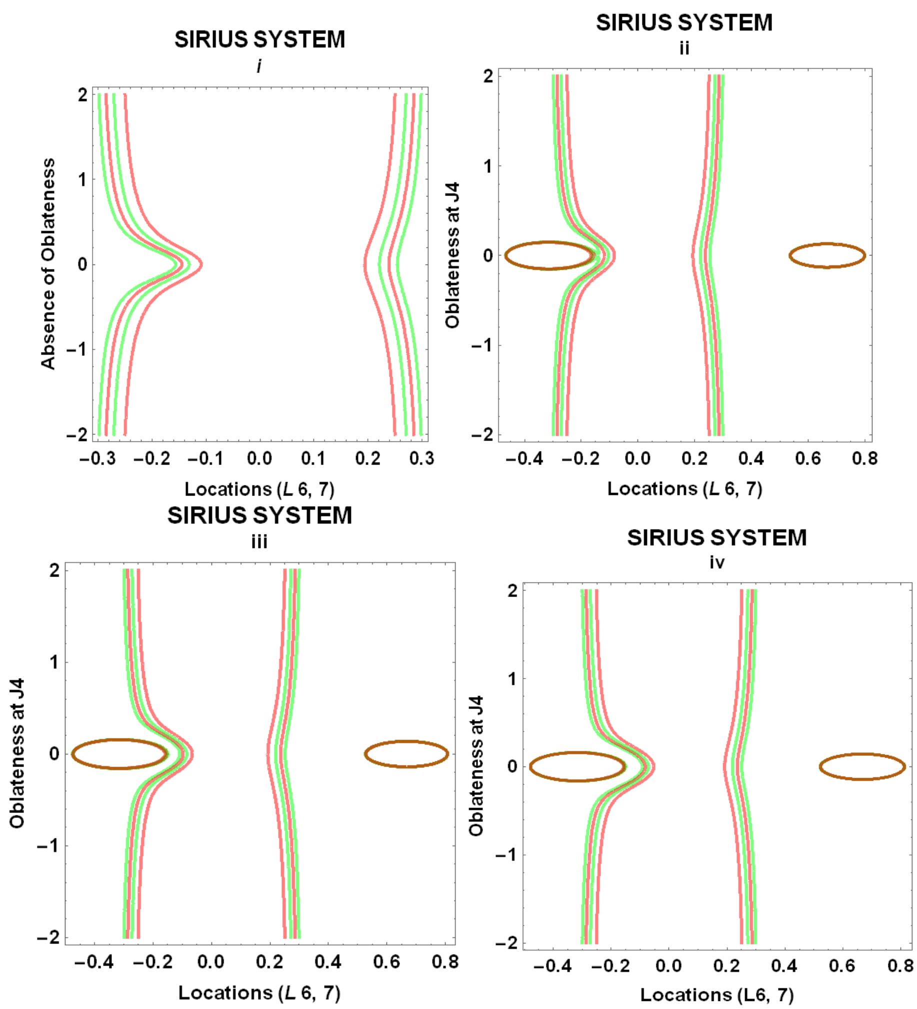


Fig. 6. Zero-Velocity Curves for Sirius system with Oblateness at J_4 (A_2 & B_2)

6. NUMERICAL APPLICATIONS

Considering (14), (17) and (29), the locations and stability of the out-of-plane libration points are computed numerically using the software package MATHEMATICA for the systems Gliese 667 and Sirius. Tables 2-5 present the effects due to the perturbed oblateness up to zonal harmonics J_4, semi-major axis and eccentricity of the obits on the location and stability of the stated problem. We considered $a = 1 - \alpha$ and $\alpha \ll 1$ in the computation. Using software package GNU-plots, the effects of these parameters on the positions of out-of-plane libration points are demonstrated graphically in Figs. 7-10.

Table 1. Numerical data

Binary system	Masses		Mass ratio (μ)	Semi-major axis (a)	Eccentricity (e)
	m_1	m_2			
Gliese 667	0.73	0.31	0.2981	0.2500	0.5800
Sirius	2.02	0.978	0.3262	2.8582	0.5942

Table 2. Stability and locations of out-of-plane libration points for Gliese 667 for $e = 0.1$

Oblateness				Positions of out-of-plane points		Stability of out-of-plane points		
A_1	A_2	B_1	B_2	x	$\pm z$	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0.00	0.00	0.0	0.00	-0.2981	0.4177	$\pm 0.4250i$	$\pm 1.0493i$	$\pm 2.3736i$
0.01	-0.005	0.02	-0.01	-0.3338	0.4456	$\pm 0.3963i$	$\pm 1.0400i$	$\pm 2.3612i$
0.02	-0.01	0.04	-0.02	-0.3694	0.4718	$\pm 0.3631i$	$\pm 1.0308i$	$\pm 2.3490i$
0.03	-0.015	0.06	-0.03	-0.4051	0.4967	$\pm 0.3242i$	$\pm 1.0219i$	$\pm 2.3370i$
0.04	-0.02	0.08	-0.04	-0.4408	0.5203	$\pm 0.2770i$	$\pm 1.0131i$	$\pm 2.3251i$
0.05	-0.025	0.10	-0.05	-0.4765	0.5430	$\pm 0.2160i$	$\pm 1.0046i$	$\pm 2.3133i$

Table 3. Stability and locations of out-of-plane libration points for Gliese 667 with $e = 0.2$

Oblateness				Positions of out-of-plane points		Stability of out-of-plane points		
A_1	A_2	B_1	B_2	x	$\pm z$	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0.00	0.00	0.0	0.00	-0.2981	0.4177	$\pm 0.4250i$	$\pm 1.0493i$	$\pm 2.3736i$
0.01	-0.005	0.02	-0.01	-0.3365	0.4322	$\pm 0.3970i$	$\pm 1.0366i$	$\pm 2.3644i$
0.02	-0.01	0.04	-0.02	-0.3749	0.4461	$\pm 0.3645i$	$\pm 1.0242i$	$\pm 2.3554i$
0.03	-0.015	0.06	-0.03	-0.4134	0.4597	$\pm 0.3260i$	$\pm 1.0120i$	$\pm 2.3465i$
0.04	-0.02	0.08	-0.04	-0.4518	0.4728	$\pm 0.2790i$	$\pm 1.0002i$	$\pm 2.3377i$
0.05	-0.025	0.10	-0.05	-0.4902	0.4856	$\pm 0.2179i$	$\pm 0.9888i$	$\pm 2.3292i$

Table 4. Stability and locations of out-of-plane libration points for Sirius with $e = 0.1$

Oblateness				Positions of out-of-plane points		Stability of out-of-plane points		
A_1	A_2	B_1	B_2	x	$\pm z$	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0.00	0.00	0.0	0.00	-0.3262	1.0939	± 0.5527	$\pm 0.6438i$	$\pm 2.1026i$
0.01	-0.005	0.02	-0.01	-0.3466	1.0488	± 0.6206	± 0.6563□	± 2.0939□
0.02	-0.01	0.04	-0.02	-0.3670	1.0017	± 0.6805	$\pm 0.6667i$	$\pm 2.0854i$
0.03	-0.015	0.06	-0.03	-0.3874	0.9522	± 0.7349	$\pm 0.6754i$	$\pm 2.0771i$
0.04	-0.02	0.08	-0.04	-0.4079	0.9001	± 0.7851	$\pm 0.6829i$	$\pm 2.0690i$
0.05	-0.025	0.10	-0.05	-0.4283	0.8447	± 0.8321	$\pm 0.6895i$	$\pm 2.0610i$

Table 5. Stability and locations of out-of-plane libration points for Sirius with $e = 0.2$

Oblateness				Positions of out-of-plane points		Stability of out-of-plane points		
$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{B_1}$	$\mathbf{B_2}$	**x**	$\pm$**z**	$\pm\boldsymbol{\lambda}_{1,2}$	$\pm\boldsymbol{\lambda}_{3,4}$	$\pm\boldsymbol{\lambda}_{5,6}$
0.00	*0.00*	*0.0*	*0.00*	-0.3262	*1.0939*	± 0.5527	$\pm 0.6438i$	$\pm 2.1026i$
0.01	-0.005	*0.02*	-0.01	-0.3463	*1.0486*	± 0.6216	$\pm 0.6543i$	$\pm 2.0973i$
0.02	-0.01	*0.04*	-0.02	-0.3665	*1.0014*	± 0.6823	$\pm 0.6628i$	$\pm 2.0921i$
0.03	-0.015	*0.06*	-0.03	-0.3866	*0.9518*	± 0.7373	$\pm 0.6700i$	$\pm 2.0870i$
0.04	-0.02	*0.08*	-0.04	-0.4068	*0.8994*	± 0.7880	$\pm 0.6761i$	$\pm 2.0821i$
0.05	-0.025	*0.10*	-0.05	-0.4269	*0.8439*	± 0.8354	$\pm 0.6815i$	$\pm 2.0772i$

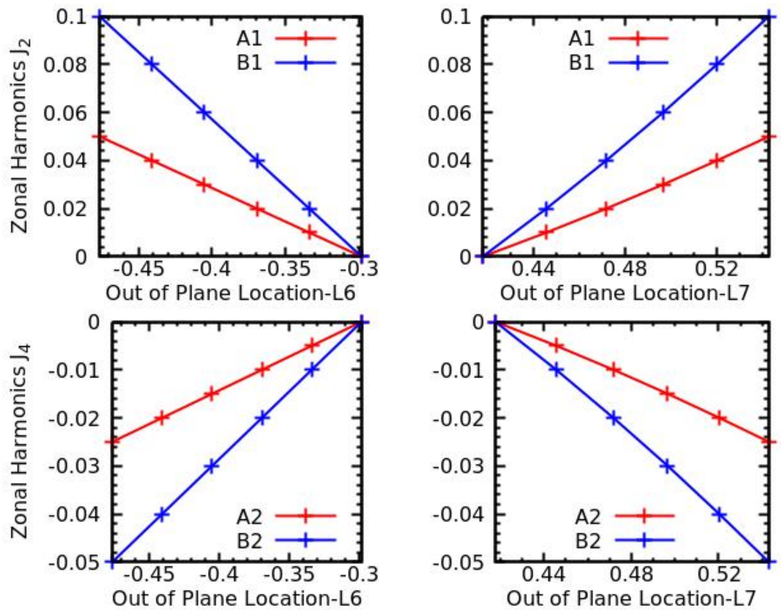


Fig. 7. Effects of oblateness on $L_{6,7}$ for Gliese 667 system for $e = 0.1$

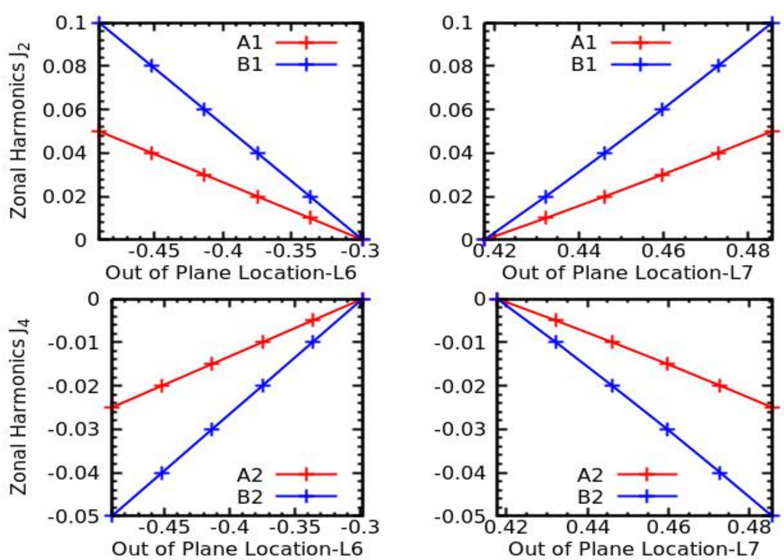


Fig. 8. Effects of oblateness on $L_{6,7}$ for Gliese 667 system for $e = 0.2$

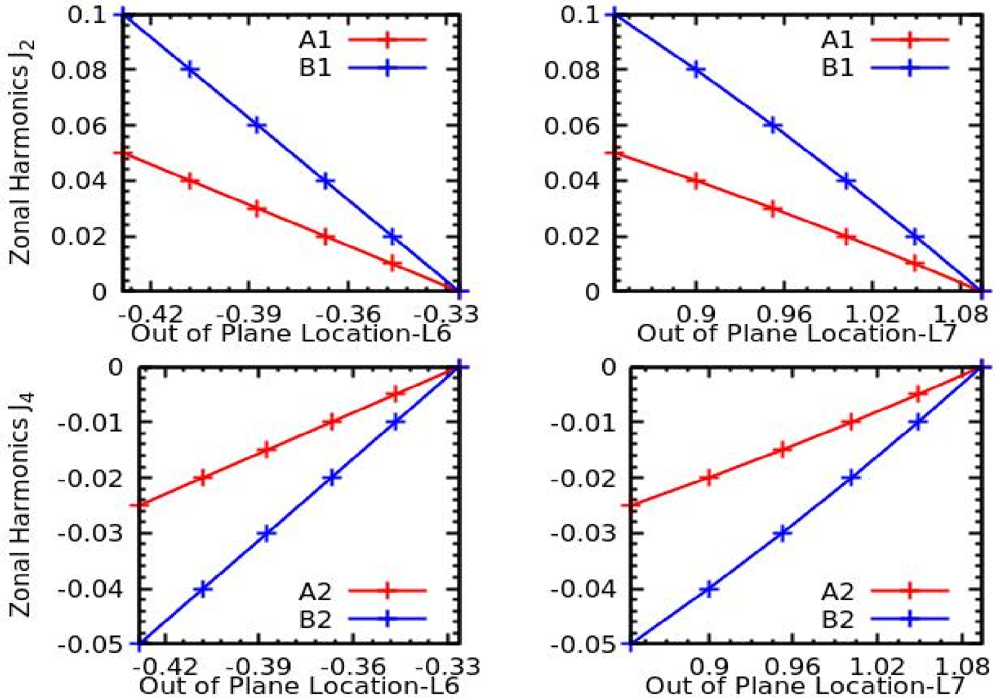


Fig. 9. Effects of oblateness on $L_{6,7}$ for Sirius system with $for\ e = 0.1$

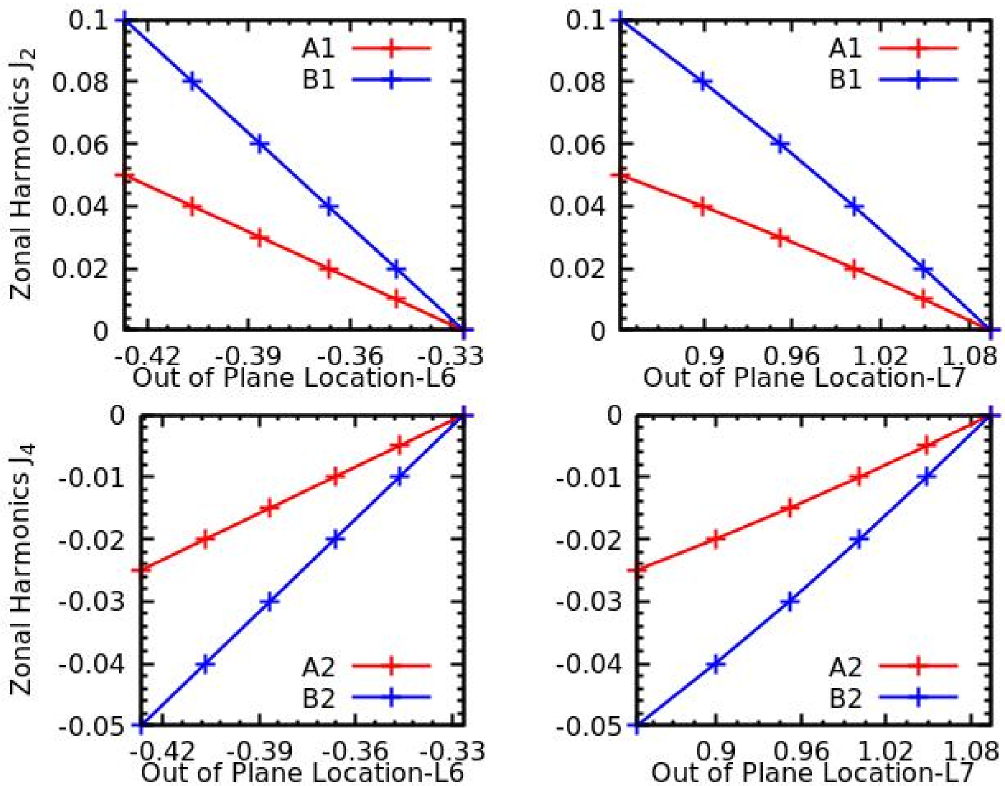


Fig. 10. Effects of oblateness on $L_{6,7}$ for Sirius system for $e = 0.2$

7. DISCUSSION AND CONCLUSION

The motion of the infinitesimal mass in the out-of-plane libration points moving in elliptic orbits J_4 in the field of stellar binary systems: Gliese 667 and Sirius is described in Equations 1-4. Equations 14 and 17 locate the positions of the out-of-plane libration points denoted by $L_{6,7}$. The results of our study agrees with [26] when the eccentricity and the Zonal harmonics up to J_4 oblateness are absent. With the help of equations 2, 3 and 4, we were able to present the Zero-Velocity Curves (ZVC) as our quantitative method in order to avail the information about the motion of the third body near the out-of-plane libration points for the binary systems considered in this study. The possible topologies of these curves are demonstrated in Figs. 3-6 for Gliese 667 and Sirius systems. As evidenced in Tables 2-5 and Figs. 7-10, the positions of the studied problem are greatly affected by the perturbed parameters involved. This has confirmed with the results of [2,19,21]. The effects of this stated parameters on the locations and the size of stability region of the study under review are shown in Tables 2-5 for the systems: Gliese 667 and Sirius. We observed that, the topologies of ZVC on the positions of the out-of-plane libration points for the binary systems: Gliese 667 and Sirius seems to move apart in the absence of oblateness $(A_1 = B_1 = A_2 = B_2 = 0)$. This is evidence in Figs. 3 & 5 of label A and Figd. 4 & 6 of label i for both binary systems. However, it seems to move closer in the presence of oblateness for both binary systems as shown in Figs. 3 & 5 of label $B - D$ and Figs. 4 & 6 of label $ii - iv$. As presented in Figs. 7-10, the positions of the out-of-plane libration points $(L_{6,7})$ of the infinitesimal body lie in the $xz-$ plane almost directly above and below the center of each oblate primary. As it is evidenced in Tables 2-5, for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane libration points are unstable when both primaries are considered for Gliese 667 and Sirius systems.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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