Asian Research Journal of Mathematics

Volume 20, Issue 9, Page 76-80, 2024; Article no.ARJOM.121703

ISSN: 2456-477X

On Flc-Focal Curves According Flc-Frame

Thiago Mendonça [a*]

[a] *Departamento de Matemática, UFRPE, Rua Manoel de Medeiros, s/n - Dois Irmãos, Recife - PE,* 52171 − 900, *Brazil.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: https://doi.org/10.9734/arjom/2024/v20i9831

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/121703

Received: 11/06/2024
Accepted: 19/08/2024
Published: 28/08/2024

Original Research Article

Abstract

In this study, we first characterize focal curves by examining the FLC-frame in three-dimensional Euclidean space. We then derive the relationship between the curvatures of a curve and the focal curvatures. Finally, we present some new conditions for curves with constant curvatures in $\mathbb{E}^3$.

Keywords: Flc-frame; focal curve; focal curvatures.

2010 Mathematics Subject Classification: 53C25, 83C05, 57N16.

1 Backround on Flc-Frame

Consider the tridimensional Euclidean space $\mathbb{E}^3$ with inner product

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2,$$

**Corresponding author: E-mail:thiago.mendonca@ufrpe.br;*

Cite as: *Mendonca, Thiago. 2024. "On Flc-Focal Curves According Flc-Frame". Asian Research Journal of Mathematics 20 (9):76-80. https://doi.org/10.9734/arjom/2024/v20i9831.*

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Consider the curve $\alpha : I \to \mathbb{E}^3$, which is differentiable in the Euclidean 3-space and is defined on an open interval I. The Frenet frame is defined as follows [1]

$$t = \frac{\alpha'}{\| \alpha' \|}, \quad b = \frac{\alpha' \wedge \alpha''}{\| \alpha' \wedge \alpha'' \|}, \quad n = b \wedge t, \tag{1.1}$$

satisfying

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \| \alpha' \| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{1.2}$$

Here, κ and τ are differentiable functions defined on I, referred to as the curvature and torsion of α, respectively. The vectors t, n and b represent the tangent, principal normal, and binormal vectors of α, respectively.

In addition to the Frenet frame, it is possible to establish other frame along a three-dimensional curve such as: Bishop frame, q-frame, alternative frame and others [2, 3, 4]. Recently, Dede [5] introduced a newframe along a polynomial space curve, called as Flc-frame. The computation of Flc-frame is easier than the Frenet frame and has some advantages, such as (see [6]):

i) There are no singular points of order 1, that is, there is no $t_0 \in I$ such that $\alpha''(t_0) = 0$, in the FLC frame, whereas the Frenet frame exhibits irregular behavior at a first-order singular point.

ii) The chances of inflection points occurring, that is, points $t_0 \in I$ such that $\alpha(t_0) \wedge \alpha''(t_0) = 0$, are lower in the FLC frame than in the Frenet frame.

Discussion of the Flc-frame and its application to the tube surfaces can be found in [5], on geometry of focal surfaces in [7], in the analysis of Smarandache ruled surfaces in [8].

Let $\alpha(t)$ be a polynomial space curve of degree n. The Flc-frame is given by

$$t = \frac{\alpha'}{\| \alpha' \|}, \quad D_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\| \alpha' \wedge \alpha^{(n)} \|}, \quad D_2 = D_1 \wedge t, \tag{1.3}$$

where the prime ' indicates the differentiation with respect to t and the notation $\alpha^{(n)}$ expresses the nth derivative of the curve α with respect to t [5]. The new vectors D_1 and D_2 are called as *binormal-like vector* and *normal-like vector*, respectively.

Calculations show that the derivatives of the Flc-frame satisfy [5]

$$\begin{bmatrix} t' \\ D_2' \\ D_1' \end{bmatrix} = \| \alpha' \| \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ D_2 \\ D_1 \end{bmatrix}, \tag{1.4}$$

where

$$d_1 = \frac{\left\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \right\rangle}{\| \alpha' \|^3 \| \alpha' \wedge \alpha^{(n)} \|}, \quad d_2 = \frac{\det(\alpha'', \alpha', \alpha^{(n)})}{\| \alpha' \|^2 \| \alpha' \wedge \alpha^{(n)} \|}, \quad d_3 = \frac{\det(\alpha', \alpha'', \alpha^{(n)}) \left\langle \alpha', \alpha^{(n)} \right\rangle}{\| \alpha' \|^2 \| \alpha' \wedge \alpha^{(n)} \|} \tag{1.5}$$

are called the curvatures of the Flc-frame.

Corollary 1.1. *If the degree of polynomial space curve is two, then the Flc-frame coincides with the Frenet frame with curvatures $d_1 = \kappa, d_2 = 0$ and $d_3 = \tau = 0$ [6].*

A focal curve or generalized evolute is the geometric locus given by the centers of the osculating circles of a given curve. With applications ranging from Dynamical Systems Theory to Surface Engineering [9], the focal curve can be expressed in terms of the frame of the initial curve and in a parametrization the coefficients called focal curvatures are obtained. These curves are studied in different spaces and frames, for example in [10, 11, 12, 13, 14, 15]. Motivated by these, in this paper, we study the focal curves according Flc-frame in $\mathbb{E}^3$.

2 Focal Curves According Flc-Frame in $\mathbb{E}^3$

Let $\alpha : I \to \mathbb{E}^3$ be a regular space curve in the three-dimensional Euclidean space $\mathbb{E}^3$ with nonzero curvature κ and torsion τ. The focal curve of α is the curve given by the equation

$$\beta(t) = \alpha(t) + \varphi_1(t)n(t) + \varphi_2(t)b(t), \tag{2.1}$$

where n is a principal unit normal vector field of α, b is a binormal unit vector field of α. The coefficients $\varphi_1(t)$ and $\varphi_2(t)$ are smooth functions called focal curvatures of α [16].

In terms of the Flc-frame, the focal curve of α is given by

$$\beta(t) = \alpha(t) + \varphi_1(t)D_2(t) + \varphi_2(t)D_1(t). \tag{2.2}$$

Theorem 2.1. *Consider a unit speed curve $\alpha : I \to \mathbb{E}^3$ and its corresponding focal curve β. Then,*

$$\begin{aligned}\beta(s) &= \alpha(s) + e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right] D_2 \\ &+ \left\{\frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right]\right\} D_1,\end{aligned} \tag{2.3}$$

where C is a constant of integration.

Proof. Suppose α is a curve with unit speed and β represents its focal curve in $\mathbb{E}^3$.

Differentiating the equation (2.2) and using (1.4), we obtain

$$\beta' = (1 - d_1\varphi_1 - d_2\varphi_2)t + (\varphi_1' - d_3\varphi_2)D_2 + (d_3\varphi_1 + \varphi_2')D_1. \tag{2.4}$$

From equation (2.4), the first two components vanish, we get

$$1 - d_1\varphi_1 - d_2\varphi_2 = 0, \tag{2.5}$$

$$\varphi_1' - d_3\varphi_2 = 0. \tag{2.6}$$

From equation (2.5),

$$\varphi_2 = \frac{1 - d_1\varphi_1}{d_2}.$$

In (2.6),

$$\varphi_1' + \frac{d_1 d_3}{d_2}\varphi_1 = \frac{d_3}{d_2}.$$

By integrating this equation, we find

$$\varphi_1 = e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right],$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right].$$

Substituting in the equation (2.2), the result follows.

□

As an immediate consequence of the above theorem, we have:

Corollary 2.2. *Consider a unit speed curve $\alpha : I \to \mathbb{E}^3$ and its corresponding focal curve β in $\mathbb{E}^3$. Then, the focal curvatures of β are*

$$\varphi_1 = e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right],$$
$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right].$$

Based on Theorem 2.1, we can state the following corollary:

Corollary 2.3. *Consider a unit speed curve $\alpha : I \to \mathbb{E}^3$ and its focal curve β in $\mathbb{E}^3$. If d_1, d_2 and d_3 are constants, then the focal curvatures of β are*

$$\varphi_1 = \frac{1}{d_1} + Ce^{-\frac{d_1 d_3}{d_2} s},$$
$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} \left(\frac{1}{d_1} + Ce^{-\frac{d_1 d_3}{d_2} s} \right).$$

3 Conclusion

The study of focal curves in the Flc-frame provides a new view of these curves, revealing new characteristics that were not evident in other frames. In this study we characterise focal curves in euclidean space according Flc-frame and then deduce relationships between the curvature of a curve and its focal curvatures.

Disclaimer (Artificial Intelligence)

Author hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Acknowledgement

We would like to thank the referees for their valuable comments and suggestions.

Competing Interests

Author has declared that no competing interests exist.

References

[1] do Carmo MP. Differential geometry of curves and surfaces, Prentice-Hall, Englewood Cliffs, NJ; 1976.

[2] Bishop RL. There is more than one way to frame a curve, Amer. Math. Monthly. 1975;82:246–251.

[3] Körpınar T, Körpınar Z, Asil V. Optical effects of some motion equations on quasi-frame with compatible Hasimoto map, Optik. 2022;251:168291.

[4] Turhan T, Topdal HT. On geometry of some curve pairs according to type of bishop frame In Lorentz 3-Space. Journal of Pharmaceutical Negative Results. 2023;354-360.

[5] Dede M. A new representation of tubular surfaces. Houston Journal of Mathematics. 2019;45(3):707-720.

[6] Dede M. Why Flc-Frame is Better than Frenet Frame on Polynomial Space Curves? Mathematical Science and Applications E-Notes. 2022;10(4):190-198.

[7] Süleyman Ş, Kebire Hilal A. On geometry of focal surfaces due to Flc frame in Euclidean 3-space, Authorea. November. 2022;11.

[8] Süleyman Ş, Hilal A, Davut C. Special Smarandache Ruled Surfaces According to Flc Frame in $\mathbb{E}^3$. Applications and Applied Mathematics: An International Journal (AAM). 2023;06.

[9] Mortenson ME. Geometric modeling, John Wiley & Sons, Inc.; 1997.

[10] Asil V, Bas S, Körpinar T. On construction of d-focal curves in euclidean 3-Space $\mathbb{M}^3$, Boletim da Sociedade Paranaense de Matemática. 2013;31(2):273-277.

[11] Asil V, Bas S, Körpinar T. New characterization of D- Focal Curves in Minkowski 3-space. Boletim da Sociedade Paranaense de Matemática. 2020;38(2):115-123.

[12] Dede M. On polynomial space curves with Flc-frame. Turkish Journal of Mathematics and Computer Science. 2023;15(2):414-422.

[13] Dinkova CL, Encheva RP. Generalized focal curves of regular C^3-plane curves. Section Mathematics and Natural Sciences. 2022;57-65.

[14] Georgiev GH, Dinkova CL. Generalized focal curves of frenet curves in three-dimensional euclidean space. Global Journal of Pure and Applied Mathematics. 2020;16(6), 891-913.

[15] Körpınar T. On quasi focal curves with quasi frame in space. Boletim da Sociedade Paranaense de Matemática. 2023;41:1-3.

[16] Gray A, Abbena E, Salamon S. Modern differential geometry of curves and surfaces. Chapman Hall/CRC; 2006.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/121703